Principal Symbols in Noncommutative Geometry

Séminaire d'Algèbres d'Opérateurs,
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Summary

1. Pseudodifferential operators and their symbols
2. Noncompact manifolds
3. Quantum ergodicity

This talk is partially based on work in progress with Galina Levitina, Edward McDonald, Fedor Sukochev, and Dmitriy Zanin.

Pseudodifferential operators and their symbols

Hamiltonian mechanics

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Given a Hamiltonian $H : T^*M \to \mathbb{R}$, the time-evolution of f is given by

$$\frac{df}{dt} = \{f, H\},$$

where in local coordinates the Poisson bracket is defined as

$$\{f, g\} := \sum_j \left(\frac{\partial f}{\partial x_j} \frac{\partial g}{\partial \xi_j} - \frac{\partial f}{\partial \xi_j} \frac{\partial g}{\partial x_j} \right).$$

Note that $\{\,\cdot\,, H\} = \sum_j \left(\frac{\partial H}{\partial \xi_j} \frac{\partial}{\partial x_j} - \frac{\partial H}{\partial x_j} \frac{\partial}{\partial \xi_j} \right)$ forms a vector field on T^*M, which is called the Hamiltonian vector field. The corresponding flow on T^*M is denoted $\Phi_H(t)$.

Quantum mechanics

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The time-evolution of a state $\xi \in \mathcal{H}$ in a Hilbert space is governed by one-parameter unitary groups $\xi(t) = e^{itH}\xi$. Equivalently, the *observables* $B \in \mathcal{L}(\mathcal{H})_{sa}$ evolve as $B(t) = e^{-itH}Be^{itH}$. It follows that

$$\frac{d}{dt}B(t) = e^{-itH}[B, H]e^{itH}.$$

Quantisation

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This is too much to ask for, but pseudodifferential operators get really close. Points 2 and 4 will at best only hold 'up to lower order operators'.

Definition (Pseudodifferential operators on $\mathbb{R}^d$)

We say that $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$, $m \in \mathbb{R}$, if $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and if

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq A_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|}, \quad \alpha, \beta \in \mathbb{N}, x, \xi \in \mathbb{R}^d,$$

here $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$. We define the operator $T_a : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$

$$T_a f(x) := \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} a(x,\xi) \hat{f}(\xi) \, d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

The class of operators T_a is denoted $\Psi^m(\mathbb{R}^d)$.

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On manifolds, we can glue such operators together (up to smoothing operators) to obtain ΨDOs on manifolds. Symbols can be defined as functions on the cotangent space T^*M.

Classical pseudodifferential operators

Let $\overline{\mathbb{R}^d}$ be the radial compactification of $\mathbb{R}^d$: we glue a 'celestial sphere' $\mathbb{S}^{d-1}$ to $\mathbb{R}^d$ at 'infinity'. That is, for the function $\rho : \mathbb{R}^d \setminus \{0\} \to \mathbb{R}$ defined by $\rho(x) = \frac{1}{|x|}$, we are adding the zero level-set to $\mathbb{R}^d$.

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More rigorously:

$$\overline{\mathbb{R}^d} := \left(\mathbb{R}^d \sqcup \left(\mathbb{S}^{d-1} \times [0,1)_\rho\right)\right) \Big/ \sim, \quad \mathbb{R}^d \ni (\theta, r) \sim (\theta, \rho) \text{ if } \rho = \frac{1}{r}.$$

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The corresponding pseudodifferential operators are denoted by $\Psi^m_{cl}(\mathbb{R}^d)$.

Classical pseudodifferential operators (II)

By Taylor's theorem, the condition that $a \in C^\infty(\mathbb{R}^d \times \overline{\mathbb{R}^d})$ is equivalent to the existence of an asymptotic expansion

$$a(x,\xi) \sim \sum_{k=0}^{\infty} a_{-k}(x,\xi), \quad a_{-k} \in S^{-k}(\mathbb{R}^d \times \mathbb{R}^d),$$

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Likewise, on manifolds we can radially compactify the fibres of T^*M, denoted by $\overline{T^*M}$. We say that $a \in S^m_{cl}(T^*M)$ is a classical symbol if $a\langle\xi\rangle^{-m}$ extends to $C^\infty(\overline{T^*M})$.

Principal symbols

For this compactification, we have a boundary

$$\partial\overline{T^*M} := \overline{T^*M} \setminus T^*M \simeq S^*M,$$

explicitly for $\mathbb{R}^d$ given by $\mathbb{R}^d \times \mathbb{S}^{d-1}$.

For a classical symbol $a \in S^0_{cl}(T^*M)$, we call the restriction

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$$[a] \in S^0_{cl}(T^*M)/S^{-1}(T^*M),$$

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NB: this story also works for different compactifications!

Sequined donut

The space S^*M is a sphere bundle on M: at each point in M the fibre is a sphere $\mathbb{S}^{d-1}$. For a two-dimensional space, this looks like sequin fabric.

Exact sequences

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In fact, we also have an exact sequence

$$0 \to \Psi^{-1}_{cl}(M) \to \Psi^0_{cl}(M) \to C^\infty(S^*M) \to 0.$$

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Of course, other quantisations exist too.

Egorov's theorem

Let $A \in \Psi^1_{cl}(M)$ with symbol $a \in S^1_{cl}(T^*M)$ be a self-adjoint elliptic classical ΨDO (elliptic meaning that $\sigma_0(A)$ is nowhere 0). Then for $B \in \Psi^0_{cl}(M)$, we have that $e^{-itA}Be^{itA} \in \Psi^0_{cl}(M)$ and

$$\sigma_0\left(e^{-itA}Be^{itA}\right) = \sigma_0(B) \circ \Phi_a(t),$$

where $\Phi_a(t)$ is the flow of the Hamiltonian vector field generated by a, i.e.

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This theorem relates the time-evolution of Hamiltonian mechanics to the time-evolution of the corresponding quantised observables.

Geodesic flow

For a Riemannian manifold (M, g), we define the geodesic flow

$$G_t : SM \to SM, \quad t \in \mathbb{R},$$

in the usual way. By duality, we can likewise define $G_t : S^*M \to S^*M$, where $S^*M \subseteq T^*M$.

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The geodesic flow G_t coincides with the flow with the Hamiltonian vector field generated from $\sqrt{\Delta_g}$ restricted to S^*M, where Δ_g is the Laplace–Beltrami operator. Namely, the vector field $\{\,\cdot\,, \|\xi\|\}$ on T^*M is locally given by $\frac{1}{\|\xi\|}\sum_j \xi_j \partial_{x_j}$. Hence,

$$\sigma_0\big(e^{-it\sqrt{\Delta_g}} B e^{it\sqrt{\Delta_g}}\big) = \sigma_0(B) \circ G_t.$$

A C^*-algebraic approach

On a compact manifold M, we have that operators in $\Psi^{-1}(M)$ extend to compact operators on $L_2(M)$. In fact,

$$K(L_2(M)) \cap \Psi^0_{cl}(M) = \Psi^{-1}_{cl}(M).$$

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The exact sequence

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can be upgraded to an exact sequence of C^*-algebras (this is not immediate)

$$0 \to K(L_2(M)) \cap \overline{\Psi^0_{cl}(M)}^{\|\cdot\|} \to \overline{\Psi^0_{cl}(M)}^{\|\cdot\|} \to C(S^*M) \to 0.$$

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Through Egorov's theorem and Stone–Weierstrass, it's not too difficult to see that we also have

$$C^*\left(\bigcup_{t\in\mathbb{R}} e^{-it\sqrt{\Delta}} C^\infty(M) e^{it\sqrt{\Delta}} + K(L_2(M))\right) / K(L_2(M)) \simeq C(S^*M).$$

Noncommutative cosphere bundle

For a (compact) spectral triple $(\mathcal{A}, \mathcal{H}, D)$, we define (Connes '96, Golse–Leichtnam '98)

$$S^*\mathcal{A} := C^*\left(\bigcup_{t\in\mathbb{R}} e^{-it|D|}\mathcal{A}e^{it|D|} + K(\mathcal{H})\right)/K(\mathcal{H}).$$

This C^*-algebra comes with automorphisms

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In the commutative case, we recover $C(S^*M)$ with its geodesic flow.

Microlocal Weyl law

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$$\mathrm{Tr}(\chi_{[0,\lambda]}(\Delta)) \sim C_d \mathrm{vol}(M)\lambda^{\frac{d}{2}}, \quad \lambda \to \infty.$$

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There exists a *local* version of Weyl's law, which gives for $f \in C^\infty(M)$,

$$\mathrm{Tr}(M_f \chi_{[0,\lambda]}(\Delta)) = \sum_{n=0}^{N(\lambda)} \langle e_n, M_f e_n \rangle \sim C_d \lambda^{\frac{d}{2}} \int_M f \, d\nu_g, \quad \lambda \to \infty.$$

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Or, even, a *microlocal* Weyl law, which states for $A \in \Psi^0(M)$,

$$\mathrm{Tr}(A\chi_{[0,\lambda]}(\Delta)) = \sum_{n=0}^{N(\lambda)} \langle e_n, A e_n \rangle \sim C_d \lambda^{\frac{d}{2}} \int_{S^*M} \sigma_0(A) \, d\mu, \quad \lambda \to \infty.$$

Connes exploited these laws to obtain an operator algebraic approach to integration.

Dixmier traces

Let $\mathcal{H}$ be a Hilbert space. An eigenvalue sequence of a compact operator $A \in K(\mathcal{H})$ is a sequence $\{\lambda(k, A)\}_{k\in\mathbb{N}}$ of the eigenvalues of A listed with multiplicity, such that $\{|\lambda(k, A)|\}_{k\in\mathbb{N}}$ is non-increasing.

The usual operator trace Tr can be characterised for trace class operators $A \in \mathcal{L}_1 \subset K(\mathcal{H})$ as

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The Dixmier trace is defined on so-called weak trace class operators $A \in \mathcal{L}_{1,\infty} \subset K(\mathcal{H})$ by

$$\mathrm{Tr}_\omega(A) = \omega\text{-}\lim_{n \to \infty} \frac{1}{\log(2+n)} \sum_{k=1}^{n} \lambda(k, A),$$

where $\omega \in \ell_\infty(\mathbb{N})^*$ is an extended limit. Note that $\mathcal{L}_1 \subset \mathcal{L}_{1,\infty}$, but if $A \in \mathcal{L}_1$, $\mathrm{Tr}_\omega(A) = 0$.

Connes' integral formula

Connes proved the following.

Connes' Integral Formula

Let (M, g) be a compact Riemannian manifold, $f \in C_c^\infty(M)$. Then for any Dixmier trace Tr_ω,

$$\mathrm{Tr}_\omega(M_f(1-\Delta_g)^{-\frac{d}{2}}) = C_d \int_M f \, d\nu_g.$$

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Or stronger, for $A \in \Psi^0_{cl}(M)$,

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Let (M, g) be a compact Riemannian manifold, $f \in C_c^\infty(M)$. Then for any Dixmier trace Tr_ω,

$$\mathrm{Tr}_\omega(M_f(1-\Delta_g)^{-\frac{d}{2}}) = C_d \int_M f\, d\nu_g.$$

Or stronger, for $A \in \Psi^0_{cl}(M)$,

$$\mathrm{Tr}_\omega(A(1-\Delta_g)^{-\frac{d}{2}}) = C_d \int_{S^*M} \sigma_0(A)\, d\mu.$$

Connes' result is in fact even stronger, as he does not assume a Riemannian structure.

Non-compact manifolds

The problem

Before thinking about non-compact spectral triples, we need to think about non-compact manifolds. For simplicity I will now just consider $\mathbb{R}^d$.

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While $C^\infty(S^*M) \simeq \Psi_{cl}^0(M)/\Psi^{-1}(M)$, it is therefore no longer true that $C(S^*M) \simeq \overline{\Psi_{cl}^0(M)}^{\|\cdot\|}/K(L_2(M))$.

Scattering Calculus

Definition (Scattering ΨDOs on $\mathbb{R}^d$)

We say that $a \in S_{sc}^{m,l}(\mathbb{R}^d \times \mathbb{R}^d)$, $m, l \in \mathbb{R}$, if $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq A_{\alpha\beta} \langle x \rangle^{l-|\beta|} \langle \xi \rangle^{m-|\alpha|}, \quad \alpha, \beta \in \mathbb{N}, x, \xi \in \mathbb{R}^d.$$

Recall that $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$.

We define $\Psi_{sc}^{m,l}(\mathbb{R}^d)$ accordingly. Note that $\Psi_{sc}^{m,0}(\mathbb{R}^d) \subsetneq \Psi^m(\mathbb{R}^d)$.

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Amazingly, $\Psi_{sc}^{m,l}(\mathbb{R}^d) \subseteq K(\mathcal{H})$ if both $m, l < 0$.

Classical scattering ΨDOs

Again we can take a shortcut to define *classical* scattering pseudodifferential operators.

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Let $\overline{\mathbb{R}^d}$ be the radial compactification of $\mathbb{R}^d$.

We define $S^{m,l}_{sc,cl}(\mathbb{R}^d \times \mathbb{R}^d) \subseteq S^{m,l}_{sc}(\mathbb{R}^d \times \mathbb{R}^d)$ as those a for which $a(x,\xi)\langle x\rangle^{-l}\langle\xi\rangle^{-d}$ extends to a smooth function $C^\infty(\overline{\mathbb{R}^d} \times \overline{\mathbb{R}^d})$.

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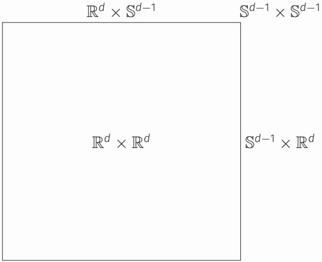
Accordingly, we define $\Psi^{m,l}_{sc,cl}(\mathbb{R}^d) \subseteq \Psi^{m,l}_{sc}(\mathbb{R}^d)$.

Note that by Taylor's theorem, this is equivalent to $a(x,\xi)$ admitting asymptotic expansions of the right kind as $x \to \infty$, as $\xi \to \infty$, and as both $x, \xi \to \infty$.

Scattering cotangent bundle

We write $\overline{{}^{sc}T^*\mathbb{R}^d} := \overline{\mathbb{R}^d} \times \overline{\mathbb{R}^d}$ for the (compactified) scattering cotangent bundle on $\mathbb{R}^d$. This is a *compact* manifold with corners, consisting of strata

$$\overline{{}^{sc}T^*\mathbb{R}^d} = (\mathbb{R}^d \times \mathbb{R}^d) \sqcup (\mathbb{S}^{d-1} \times \mathbb{R}^d) \sqcup (\mathbb{R}^d \times \mathbb{S}^{d-1}) \sqcup (\mathbb{S}^{d-1} \times \mathbb{S}^{d-1})$$



Scattering cosphere bundle

Now the principal symbol lives on $\partial\overline{{}^{sc}T^*\mathbb{R}^d}$, which consists of the strata

$$\partial(\overline{{}^{sc}T^*\mathbb{R}^d}) \simeq (\mathbb{R}^d \times \mathbb{S}^{d-1}) \sqcup (\mathbb{S}^{d-1} \times \mathbb{R}^d) \sqcup (\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}).$$

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For $A \in \Psi^{m,l}_{sc,cl}(\mathbb{R}^d)$, the equivalence class $[A] \in \Psi^{m,l}(\mathbb{R}^d)/\Psi^{m-1,l-1}(\mathbb{R}^d)$ corresponds in a natural way to a smooth function in $C^\infty(\partial \overline{^{sc}T^*\mathbb{R}^d})$.

The C^*-algebraic principal symbol

Since operators in $\Psi_{sc}^{-\varepsilon,-\varepsilon}$ are compact in the scattering calculus, we obtain the following.

Lauter–Moroianu (2001)

We have an exact sequence of C^*-algebras

$$0 \to \overline{\Psi_{sc,cl}^{0,0}}^{\|\cdot\|} \cap K(L_2(\mathbb{R}^d)) \to \overline{\Psi_{sc,cl}^{0,0}}^{\|\cdot\|} \to C(\partial\overline{^{sc}T^*\mathbb{R}^d}) \to 0.$$

Examples

For $f \in C_c^\infty(\mathbb{R}^d)$, we have $M_f(1+\Delta)^{-\frac{d}{2}} \in \Psi_{sc,cl}^{-d,-\infty}(\mathbb{R}^d)$, we have $M_f(1+\Delta)^{-\frac{d}{2}} \in \mathcal{L}_{1,\infty}$, and

$$\mathrm{Tr}_\omega(M_f(1+\Delta)^{-\frac{d}{2}}) = \frac{\mathrm{vol}\,\mathbb{S}^{d-1}}{d(2\pi)^d} \int_{\mathbb{R}^d} f(x)\,dx.$$

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Note that the $(-d,-d)$-principal symbol of $M_f(1+\Delta)^{-\frac{d}{2}}$ is a function on

$$(\mathbb{R}^d \times \mathbb{S}^{d-1}) \sqcup (\mathbb{S}^{d-1} \times \mathbb{R}^d) \sqcup (\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}),$$

and in this case it is

$$f(x) \sqcup 0 \sqcup 0.$$

The formula above is the integral of this!

Examples (II)

For the Fourier transform of $M_f(1+\Delta)^{-\frac{d}{2}}$, which is $f(\nabla)M_{\langle x\rangle^{-d}}$, we likewise have $f(\nabla)M_{\langle x\rangle^{-d}} \in \Psi^{-\infty,-d}_{sc,cl}(\mathbb{R}^d)$, and

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In this case, the principal symbol is

$$0 \sqcup f(\xi) \sqcup 0.$$

Again, we obtain the integral of this.

Known result

One might think that $\Psi^{-d,-d}_{sc,cl}(\mathbb{R}^d) \subseteq \mathcal{L}_{1,\infty}$, but this is not true:

$$M^{-d}_{\langle x \rangle}(1+\Delta)^{-\frac{d}{2}} \notin \mathcal{L}_{1,\infty}.$$

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Theorem (Nicola 2003)

Let $P \in \Psi_{sc,cl}^{-d,-d-1}(\mathbb{R}^d)$. Then $P \in \mathcal{L}_{1,\infty}$, and

$$\mathrm{Tr}_\omega(P) = \frac{1}{d(2\pi)^d} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \sigma_{sc}^{-d,-d}(P)\, d\mu.$$

If $P \in \Psi_{sc,cl}^{-d-1,-d}(\mathbb{R}^d)$, the same formula holds with integral over $\mathbb{S}^{d-1} \times \mathbb{R}^d$.

If $P \in \Psi_{sc,cl}^{-d,-d}(\mathbb{R}^d)$, then

$$\lim_{N\to\infty} \frac{1}{\big(\log(N+2)\big)^2} \sum_{n=0}^{N} \lambda(n,P) = \int_{\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}} \sigma_{sc}^{-d,-d}(P)\, d\mu.$$

New result

Theorem (H.–Levitina–McDonald–Sukochev–Zanin, WIP)

Let $P \in \Psi^{-d,-d}_{sc,cl}(\mathbb{R}^d)$. Then $P \in \mathcal{L}_{1,\infty}$ if and only if $\sigma_{sc}^{-d,-d}(P) \in L_1(\partial\overline{T^*_{sc}\mathbb{R}^d})$, in which case

$$\mathrm{Tr}_\omega(P) = \frac{1}{d(2\pi)^d}\int_{\partial\overline{T^*_{sc}\mathbb{R}^d}} \sigma_{sc}^{-d,-d}(P)\,d\mu.$$

This is a variant on known spectral asymptotics by Battisti–Coriasco (2011).

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This is a variant on known spectral asymptotics by Battisti–Coriasco (2011).

Note: if $P \in \mathcal{L}_{1,\infty} \cap \Psi^{-d,-d}_{sc,cl}(\mathbb{R}^d)$, then its principal symbol is zero at the corner $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$. The relevant part of the measure $d\mu$ here is the Lebesgue measure on $\mathbb{R}^d \times \mathbb{S}^{d-1} \sqcup \mathbb{S}^{d-1} \times \mathbb{R}^d$.

Scattering cotangent bundle

Moving to more general non-compact manifolds, we can mimic the construction of $\overline{{}^{sc}T^*\mathbb{R}^d} = \overline{\mathbb{R}^d} \times \overline{\mathbb{R}^d}$. Recall that we defined $\overline{\mathbb{R}^d}$ via a boundary defining function $\rho(x) = \frac{1}{|x|}$.

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Take now a compact manifold M with boundary, and consider a boundary defining function ρ such that $\rho > 0$ on the interior M°, and $d\rho \neq 0$ at ∂M.

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Take now a compact manifold M with boundary, and consider a boundary defining function ρ such that $\rho > 0$ on the interior M°, and $d\rho \neq 0$ at ∂M.

Scattering (co)tangent bundle

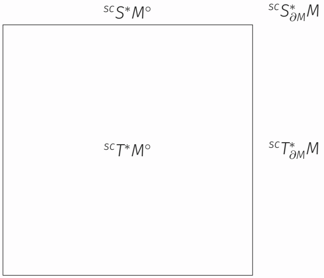
Let $(y_1, \ldots, y_n)$ be a local coordinate system of ∂M near $p \in \partial M$. Then $(\rho, y_1, \ldots, \rho_n)$ forms a coordinate system of M near p. Let $\mathcal{V}_{sc}(M)$ be the vector fields that are at every $p \in \partial M$ the $C^\infty(M)$-span of

$$x^2\partial_x, x\partial_{y_1}, \ldots, x\partial_{y_n}.$$

This defines a vector bundle ${}^{sc}TM$, and a dual ${}^{sc}T^*M$.

Melrose square

The vector bundle ${}^{sc}T^*M$ has fibres $\mathbb{R}^n$, which can be radially compactified. This results in a compact space $\overline{{}^{sc}T^*M}$, which looks like



Quantum Ergodicity

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Quantum Ergodicity

A positive self-adjoint operator Δ on $L_2(M)$ with compact resolvent, where M is a compact Riemannian manifold, is said to be **quantum ergodic** if for every orthonormal basis $\{e_n\}_{n=0}^{\infty}$ of $L_2(M)$ consisting of eigenfunctions of Δ, there exists a density one subsequence $J \subseteq \mathbb{N}$ such that

$$\lim_{J \ni j \to \infty} \langle e_j, \mathrm{Op}(a) e_j \rangle_{L_2(M)} \to \frac{1}{\mathrm{vol}(S^*M)} \int_{S^*M} a_0 \, d\mu, \quad \mathrm{Op}(a) \in \Psi^0_{cl}(M),$$

where ν is a probability measure on S^*M. In this context, a density one subsequence means that

$$\frac{\#(J \cap \{0, \ldots, n\})}{n+1} \to 1, \quad n \to \infty.$$

Pictures

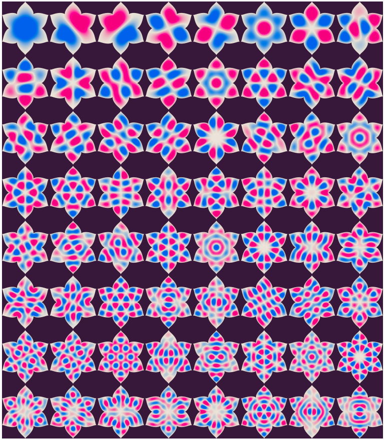

Figure 1: Eigenfunctions of the Laplacian on a rose-shaped domain, quantum ergodicity **unknown**.

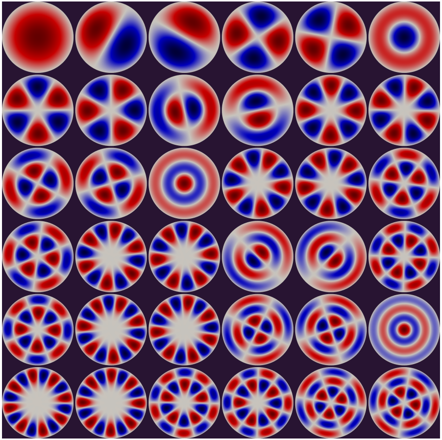

Figure 2: Eigenfunctions of the Laplacian on the disc, **not** quantum ergodic.

Pictures

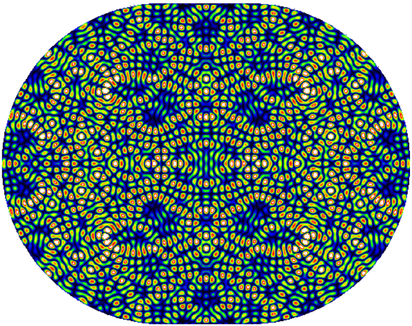

Figure 3: Typical eigenfunction of the Laplacian on a stadium, **proven** to be quantum ergodic! Credit of the picture to Douglas Stone.

Fundamental theorem of QE

The fundamental theorem that started the field of Quantum Ergodicity is the following.

Theorem (Shnirelman 1974, Zelditch 1987, Colin de Verdière 1985)

Let M be a compact Riemannian manifold. If the geodesic flow on M is ergodic, then the Laplace–Beltrami operator Δ_g is quantum ergodic.

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By now, various extensions of this theorem exist. The common thread is to study geodesic flow, and translate this into asymptotic behaviour of eigenfunctions of an operator.

We can interpret Quantum Ergodicity as a stronger microlocal Weyl law. Namely, the QE property

$$\lim_{J \ni j \to \infty} \langle e_j, A e_j \rangle_{L_2(M)} \to \frac{1}{\mathrm{vol}(S^*M)} \int_{S^*M} \sigma_0(A)\, d\mu, \quad A \in \Psi^0(M),$$

is equivalent by the Koopman–von Neumann lemma to

$$\lim_{N \to \infty} \frac{1}{N+1} \sum_{n=0}^{N} \left| \langle e_n, A e_n \rangle - \frac{1}{\mathrm{vol}(S^*M)} \int_{S^*M} \sigma_0(A)\, d\mu \right| = 0, \quad A \in \Psi^0(M).$$

QE as a Weyl law

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This is now recognisable as a stronger version of the microlocal Weyl law

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{n=0}^{N} \langle e_n, Ae_n\rangle - \frac{1}{\mathrm{vol}(S^*M)} \int_{S^*M} \sigma_0(A)\, d\mu = 0, \quad A \in \Psi^0(M).$$

Comparison

Now compare the microlocal Weyl law

$$\frac{\mathrm{Tr}(P_\lambda A P_\lambda)}{\mathrm{Tr}(P_\lambda)} \xrightarrow{\lambda\to\infty} \frac{1}{\mathrm{vol}(S^*M)} \int_{S^*M} \sigma_0(A)\, d\mu,$$

with Connes' formula

$$\mathrm{Tr}_\omega(A(1-\Delta)^{-\frac{d}{2}}) = C_d \int_{S^*M} \sigma_0(A)\, d\mu.$$

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$$\mathrm{Tr}_\omega(A(1-\Delta)^{-\frac{d}{2}}) = C_d \int_{S^*M} \sigma_0(A)\, d\mu.$$

H.–McDonald

Let $\mathcal{H}$ be a separable Hilbert space, $A \in B(\mathcal{H})$, D self-adjoint with compact resolvent, $P_\lambda := \chi_{[-\lambda,\lambda]}(D)$. If D satisfies Weyl's law $\lambda(k, |D|) \sim Ck^{\frac{1}{d}}$, then for all Dixmier traces Tr_ω,

$$\frac{\mathrm{Tr}_\omega(A(1+D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1+D^2)^{-\frac{d}{2}})} = \omega \circ M\left(\frac{\mathrm{Tr}(P_{\lambda_n} A P_{\lambda_n})}{\mathrm{Tr}(P_{\lambda_n})}\right).$$

Here, $M : \ell_\infty \to \ell_\infty$ is the logarithmc averaging defined by $M(x)_n = \frac{1}{\log(n+2)} \sum_{k=0}^{n} \frac{x_k}{k}$.

Truncated Spectral Triples

If we have a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, lots of noncommutative geometers are interested in truncated triples $(P_\lambda A P_\lambda, P_\lambda \mathcal{H}, P_\lambda D)$ (e.g. Connes–van Suijlekom, D'Andrea–Lizzi–Martinetti).

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Our result shows that if $(\mathcal{A}, \mathcal{H}, D)$ is d-dimensional and D satisfies Weyl's law, then

$$P_\lambda A P_\lambda \mapsto \frac{\mathrm{Tr}(P_\lambda A P_\lambda)}{\mathrm{Tr}(P_\lambda)}$$

is a reasonable approximation of the noncommutative integral $\mathrm{Tr}_\omega(A(1+D^2)^{-\frac{d}{2}})$.

NCG ergodicity

Since ergodicity of the geodesic flow is a measure theoretic statement, we need to take one more step.

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L_2-cosphere bundle

Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular spectral triple where D satisfies Weyl's law. Then

$$\tau(A + K(\mathcal{H})) = \frac{\mathrm{Tr}_\omega(A(1 + D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1 + D^2)^{-\frac{d}{2}})}, \quad A + K(\mathcal{H}) \in S^*\mathcal{A},$$

defines a finite positive trace on $S^*\mathcal{A}$. Then define $L_2(S^*\mathcal{A})$ as the Hilbert space of the GNS representation of $S^*\mathcal{A}$ corresponding to τ.

The geodesic flow σ_t on $S^*\mathcal{A}$ descends to a unitary operator on $L_2(S^*\mathcal{A})$.

We can now naively put forward a definition of ergodic geodesic flow for spectral triples. Namely, we say that the geodesic flow σ_t is ergodic on $(\mathcal{A}, \mathcal{H}, D)$ if the only σ_t-invariant element of $L_2(S^*\mathcal{A})$ is the identity.

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NCG QE (H.–McDonald)

Let $(\mathcal{A}, \mathcal{H}, D)$ be a d-summable regular spectral triple where D satisfies Weyl's law, and with local Weyl laws. If the geodesic flow on $(\mathcal{A}, \mathcal{H}, D)$ is ergodic, then D is quantum ergodic. That is, for every basis $\{e_n\}_{n=0}^{\infty}$ of $\mathcal{H}$ consisting of eigenvectors of D, there exists a density one subset $J \subseteq \mathbb{N}$ such that

$$\lim_{J \ni j \to \infty} \langle e_j, a e_j \rangle = \frac{\mathrm{Tr}_\omega(a(1+D^2)^{-\frac{d}{2}})}{\mathrm{Tr}_\omega((1+D^2)^{-\frac{d}{2}})}, \quad a \in \mathcal{A}.$$

Thank you!

Thanks for listening!