The Sound of the Scattering Cosphere Bundle

Analysis Seminar
University of Melbourne

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August 14 2025

Summary of this talk

1. Noncommutative geometry and Connes' trace theorem
2. Pseudodifferential operators and Connes' trace theorem (again)
3. Scattering calculus and Connes' trace theorem (again again)

This talk is based on joint work with Galina Levitina (ANU), Ed McDonald (Penn State), Fedor Sukochev (UNSW), and Dmitriy Zanin (UNSW).

Part 1: Connes' Trace Theorem

Spectral geometry

Can one hear the shape of a drum?

Figure: Mark Kac, Center for Nonlinear Studies.

Hearing the shape of a drum

The sounds a (Riemannian, compact, orientable) manifold (X, g) produces if it were a drum, correspond to the eigenvalues of the Laplace–Beltrami operator Δ_g, the manifold equivalent of the differential operator $-\sum_{j=1}^{n} \partial_{x_j}^2$.

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In other words, they correspond to those $\lambda \in \mathbb{C}$ for which the PDE

$$\begin{cases} \Delta u &= \lambda u \quad \text{on } X; \\ u|_{\partial X} &= 0 \end{cases}$$

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The question asks whether we can reconstruct our Riemannian manifold X from the data $(L_2(X), \Delta_g)$, in particular, from these eigenvalues of the operator Δ_g (this is called *spectral geometry*).

What we can hear

Weyl's law (Weyl 1910s, Duistermaat–Guillemin 1975, Ivrii 1980)

Let (X, g) be a compact Riemannian manifold of dimension d satisfying some weak assumptions. Let $N(t)$ be the number of eigenvalues (counting multiplicities) of Δ_g with absolute value less than t. Then as $t \to \infty$,

$$N(t) = \frac{\omega_d}{(2\pi)^d} \operatorname{Vol}(X) t^{\frac{d}{2}} - \frac{\omega_{d-1}}{4(2\pi)^{d-1}} \operatorname{Area}(\partial X) t^{\frac{d-1}{2}} + o(t^{\frac{d-1}{2}}).$$

What we can hear 2

Heat trace expansion (Minakshisundaram–Pleijel 1949)

We have

$$\mathrm{Tr}(\exp(t\Delta_g)) \sim \sum_{k=0}^{\infty} a_k(\Delta_g) t^{\frac{k-d}{2}}, \quad t \to 0,$$

the first coefficients of which can be given as

$$\begin{aligned} a_0(\Delta_g) &= (4\pi)^{-\frac{d}{2}} \mathrm{Vol}(X); \\ a_2(\Delta_g) &= -\frac{1}{6}(4\pi)^{-\frac{d}{2}} \int_X R \, d\mathrm{vol}_g, \end{aligned}$$

where R is the scalar curvature of X. The coefficients $a_k(\Delta_g)$ vanish for odd k, and the higher coeffients $a_k(\Delta_g)$ are integrals over X of (complicated) expressions involving the metric of X.

What we cannot hear

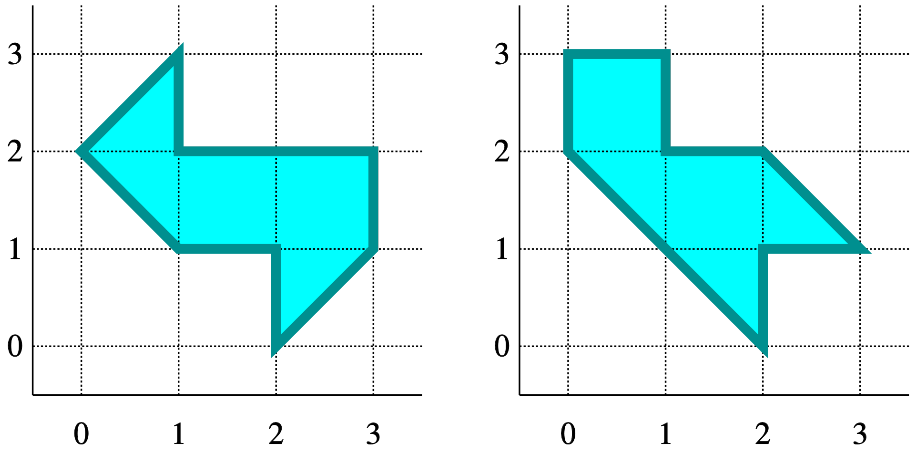

Figure: Isospectral drums.

Gelfand duality

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Where Δ_g is an operator on $L_2(X)$, the space $C(X)$ can also be represented on $L_2(X)$ via (pointwise) multiplication operators,

$$\begin{aligned} M_f : L_2(X) &\to L_2(X), \quad f \in C(X), \\ g &\mapsto fg. \end{aligned}$$

A spectral invariant

The data $(C(X), L_2(X), \Delta_g)$ is an invariant for X.

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Theorem (follows from Arendt–Biegert–ter Elst 2012)

Let (M_1, g_1) and (M_2, g_2) be connected compact Riemannian manifolds, with corresponding Laplace–Beltrami operators Δ_1 and Δ_2. Then the following are equivalent:

1. the Riemannian manifolds (M_1, g_1) and (M_2, g_2) are isometric;
2. there exists a unital $*$-isomorphism $\psi : C(M_1) \xrightarrow{\sim} C(M_2)$ and a unitary operator $U : L_2(M_1) \to L_2(M_2)$ such that

$$UM_f = M_{\psi(f)}U, \quad f \in C(M_1)$$
$$U\Delta_1 = \Delta_2 U.$$

Reconstruction theorem

We can do even better if we replace $C(M)$ by $C^\infty(M)$, and replace Δ_g by its square root D_S, a Dirac operator.

Connes' reconstruction theorem (2013)

Let $(\mathcal{A}, \mathcal{H}, D)$ be such that:

1. $\mathcal{H}$ is a Hilbert space;
2. $\mathcal{A}$ is a commutative $*$-algebra represented as bounded operators on $\mathcal{H}$;
3. D is a self-adjoint operator on $\mathcal{H}$ with compact resolvent;
4. $[D, a]$ extends to a bounded operator for all $a \in \mathcal{A}$;
5. some more technical assumptions.

Then we can construct $S \to X$ such that $(\mathcal{A}, \mathcal{H}, D) \simeq (C^\infty(X), L_2(X, S), D_S)$, where D_S is the Dirac operator on the spinor bundle $S \to X$.

Connes' integration formula

This characterisation of (spin) manifolds suggests that it should be possible to do geometry using just the multiplication operators M_f and the Laplace operator Δ_g (or the Dirac operator D_S).

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Connes' trace theorem (1988), version 1

Let (X, g) be a d-dimensional compact Riemannian manifold. Then

$$\lim_{N\to\infty} \frac{1}{\log(2+N)} \sum_{n=0}^{N} \lambda(n, M_f(1+\Delta_g)^{-\frac{d}{2}}) = C_d \int_X f \, d\mathrm{vol}_g, \quad f \in C^\infty(X).$$

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This is like a Weyl law. For the spectral counting function of $M_f(1+\Delta_g)^{-\frac{d}{2}}$, the statement is equivalent to

$$N(\lambda) = C_d \int_X f \, d\mathrm{vol}_g \cdot \lambda + o(\lambda).$$

Technical stuff

Let $\mathcal{H}$ be a Hilbert space. An eigenvalue sequence of a compact operator $A \in B(\mathcal{H})$ is a sequence $\{\lambda(k, A)\}_{k\in\mathbb{N}}$ of the eigenvalues of A listed with multiplicity, such that $\{|\lambda(k, A)|\}_{k\in\mathbb{N}}$ is non-increasing.

The usual operator trace Tr can be characterised for trace class operators $A \in \mathcal{L}_1 \subseteq B(\mathcal{H})$ as

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The Dixmier trace is defined on $\mathcal{L}_{1,\infty}$ by

$$\mathrm{Tr}_\omega(A) = \omega\text{-}\lim_{n\to\infty} \frac{1}{\log(2+n)} \sum_{k=1}^{n} \lambda(k, A), \quad A \in \mathcal{L}_{1,\infty},$$

where $\omega \in \ell_\infty(\mathbb{N})^*$ is an extended limit. Note that $\mathcal{L}_1 \subset \mathcal{L}_{1,\infty}$, but if $A \in \mathcal{L}_1$, $\mathrm{Tr}_\omega(A) = 0$.

Connes' integration formula revisited

Connes' trace theorem (1988), version 2

Let (X, g) be a d-dimensional compact Riemannian manifold. Then for any smooth function f, have $M_f(1-\Delta_g)^{-\frac{d}{2}} \in \mathcal{L}_{1,\infty}$, and for any extended limit $\omega \in (\ell_\infty)^*$,

$$\mathrm{Tr}_\omega(M_f(1-\Delta_g)^{-\frac{d}{2}}) = C_d \int_X f \, d\mathrm{vol}_g, \quad f \in C^\infty(X).$$

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This version is of deep philosophical importance to Noncommutative Geometry. In NCG, we study triples $(\mathcal{A}, \mathcal{H}, D)$ as we have seen before, but where $\mathcal{A}$ is noncommutative. There,

$$a \mapsto \mathrm{Tr}_\omega(a(1+D^2)^{-\frac{d}{2}}), \quad a \in \mathcal{A},$$

is taken as the definition of the 'noncommutative integral'.

Part 2: Pseudodifferential Operators and Connes' Trace Theorem (again)

PDEs

Suppose we want to solve the PDE (on $\mathbb{R}^2$)

$$\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial x \partial y} + \frac{\partial u}{\partial x} - 2\frac{\partial u}{\partial y} - u = f,$$

for some nice function f.

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for some nice function f.
After taking a Fourier transform, we have

$$((2\pi i\xi_1)^2 + 3(2\pi i\xi_2)^2 + (2\pi i\xi_1)(2\pi i\xi_2) + (2\pi i\xi_1) - 2(2\pi i\xi_2) + 1)\hat{u}(\xi_1, \xi_2) = \hat{f}(\xi_1, \xi_2).$$

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After taking a Fourier transform, we have

$$p_L(\xi_1, \xi_2)\hat{u}(\xi_1, \xi_2) = \hat{f}(\xi_1, \xi_2).$$

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Since $p_L(\xi_1, \xi_2) \neq 0$,

$$\hat{u}(\xi_1, \xi_2) = \frac{1}{p_L(\xi_1, \xi_2)}\hat{f}(\xi_1, \xi_2),$$

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Since $p_L(\xi_1, \xi_2) \neq 0$,

$$u(x, y) = \mathcal{F}^{-1}\left(\frac{1}{p_L(\xi_1, \xi_2)}\hat{f}(\xi_1, \xi_2)\right)(x, y),$$

solved!

Observations

- For a linear differential operator $L = \sum_{|\alpha| \leq k} a_\alpha \partial^\alpha$ on $\mathbb{R}^d$, we can write $L = \mathcal{F}^{-1} \circ M_{p_L} \circ \mathcal{F}$ where M_{p_L} indicates multiplying with the polynomial $p_L(\xi) := \sum_{|\alpha| \leq k} a_\alpha (2\pi i \xi)^\alpha$. This polynomial is called the *symbol* of L.

$$Lu(x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} p_L(\xi) \hat{u}(\xi) d\xi.$$

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$$Lu(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} p_L(\xi)\hat{u}(\xi)d\xi.$$

- If p_L is invertible, we have

$$L^{-1}u(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \frac{1}{p_L(\xi)}\hat{u}(\xi)d\xi.$$

Generalising

- Further, if $L = \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha$, we write $p_L(x,\xi) := \sum_{|\alpha|\leq k} a_\alpha(x)(2\pi i\xi)^\alpha$, and
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$$Lu(x) = \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} p_L(x,\xi)\hat{u}(\xi)d\xi.$$
- Even in this case, if $p(x,\xi) \neq 0$,
$$u \mapsto \int_{\mathbb{R}^d} e^{2\pi i x\cdot\xi} \frac{1}{p_L(x,\xi)}\hat{u}(\xi)d\xi$$
is often a pretty good guess for the inverse of L.

PsDOs

Definition (Pseudodifferential operators on $\mathbb{R}^d$)

We say that $a \in S^m(\mathbb{R}^d \times \mathbb{R}^d)$, $m \in \mathbb{R}$, if $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and if

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq A_{\alpha\beta} \langle \xi \rangle^{m-|\alpha|}, \quad \alpha, \beta \in \mathbb{N}, x, \xi \in \mathbb{R}^d,$$

here $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$. We define the operator $T_a : \mathcal{S}(\mathbb{R}^d) \to \mathcal{S}(\mathbb{R}^d)$

$$T_a f(x) := \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} a(x,\xi) \hat{f}(\xi) d\xi, \quad f \in \mathcal{S}(\mathbb{R}^d).$$

The class of operators T_a is denoted $\Psi^m(\mathbb{R}^d)$.

Principal part of differential operator

A lot of important properties of a differential operator $L = \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha$ depend on its principal part $\sum_{|\alpha|=k} a_\alpha(x)\partial^\alpha$.

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To mimic this for PsDOs, note that we can ‘see’ the principal part of L in its symbol p_L by observing that

$$p_L(x, t\xi) = \sum_{|\alpha|\leq k} t^{|\alpha|} a_\alpha(x)\xi^\alpha, \quad t > 0.$$

The principal part is the part of p_L that scales as t^k.

Classical PsDOs

We therefore define *classical* pseudodifferential operators as follows.

Definition (classical PsDOs on $\mathbb{R}^d$)

We define $S^m_{cl}(\mathbb{R}^d \times \mathbb{R}^d) \subseteq S^m(\mathbb{R}^d \times \mathbb{R}^d)$ as those a for which

$$a(x,\xi) \sim \sum_{k=0}^{\infty} a_{m-k}(x,\xi),$$

where $a_{m-k} \in S^{m-k}(\mathbb{R}^d \times \mathbb{R}^d)$ and

$$a_{m-k}(x,t\xi) = t^{m-k} a_{m-k}(x,\xi), \quad t \geq 1, |\xi| \geq 1.$$

Accordingly, we define $\Psi^m_{cl}(\mathbb{R}^d) \subseteq \Psi^m(\mathbb{R}^d)$.

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For $A \in \Psi^m_{cl}(\mathbb{R}^d)$, the equivalence class $[A] \in \Psi^m(\mathbb{R}^d)/\Psi^{m-1}(\mathbb{R}^d)$ corresponds in a natural way to the highest term in the expansion, which can be identified with a function on $\mathbb{R}^d \times \mathbb{S}^{d-1}$. This is called the *principal symbol*.

Connes trace theorem again

The construction of pseudodifferential operators so far can be performed on compact manifolds (without Riemannian structure!). The principal symbol is then a function on S^*X, the cosphere bundle, which locally looks like $U \times \mathbb{S}^{d-1}$, for $U \subseteq X$.

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Connes' trace theorem (1988), version 3

Let X be a compact manifold, and let $P \in \Psi_{cl}^{-d}(X)$. Then $P \in \mathcal{L}_{1,\infty}$, and for any extended limit $\omega \in (\ell_\infty)^*$,

$$\mathrm{Tr}_\omega(P) = \frac{1}{d(2\pi^d)} \int_{S^*X} \sigma_{-d}(P)\, d\mu.$$

Here, $\sigma_{-d}(P)$ is the part of the symbol of P that is $-d$-homogeneous, i.e. the principal symbol of P.

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For the connoisseurs: the right hand side is the Wodzicki residue.

Part 3: Scattering Calculus and Connes' Trace Theorem (again again)

Connes trace theorem on Euclidean space

For $\mathbb{R}^d$, we get the following version for free.

Connes' trace theorem (1988), version 3b

Let $P \in \Psi_{cl}^{-d}(\mathbb{R}^d)$, such that its symbol is compactly supported in the first variable. Then for any extended limit $\omega \in (\ell_\infty)^*$,

$$\mathrm{Tr}_\omega(P) = \frac{1}{d(2\pi^d)} \int_{\mathbb{R}^d \times \mathbb{S}^{d-1}} \sigma_{-d}(P)(x, \xi)\, dx d\xi.$$

Here, $\sigma_{-d}(P)$ is the part of the symbol of P that is $-d$-homogeneous, i.e. the principal symbol of P.

A Problem

This version of Connes' Trace Theorem cannot hold for all $\Psi_{cl}^{-d}(\mathbb{R}^d)$. For $g \in C_c^\infty(\mathbb{R}^d)$, define

$$P := g(\nabla)M_{\langle x\rangle^{-d}}.$$

We have that $P \in \Psi_{cl}^{-\infty}(\mathbb{R}^d) \subseteq \Psi_{cl}^{-d}(\mathbb{R}^d)$, where $\sigma_{-d}(P) = 0$.

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We have that $P \in \Psi^{-\infty}_{cl}(\mathbb{R}^d) \subseteq \Psi^{-d}_{cl}(\mathbb{R}^d)$, where $\sigma_{-d}(P) = 0$.

However, $\mathcal{F}_x \circ P \circ \mathcal{F}_x^{-1} = M_g (1+\Delta)^{-\frac{d}{2}} \in \Psi^{-d}_{cl}(\mathbb{R}^d)$ with symbol compactly supported in the first variable, and

$$\mathrm{Tr}_\omega(P) = \mathrm{Tr}_\omega(\mathcal{F}_x \circ P \circ \mathcal{F}_x^{-1}) = \frac{\mathrm{vol}\, \mathbb{S}^{d-1}}{d(2\pi)^d} \int_{\mathbb{R}^d} g(x)\, dx.$$

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Whereas on compact manifolds we have that $\Psi^{-d-1}(X) \subseteq \mathcal{L}_1 \subseteq \ker(\mathrm{Tr}_\omega)$, for non-compact spaces there can be non-trivial contributions solely from the spatial asymptotics of the symbol of the operator.

Scattering Calculus

Definition (Scattering pseudodifferential calculus on $\mathbb{R}^d$)

We say that $a \in S^{m,l}_{sc}(\mathbb{R}^d \times \mathbb{R}^d)$, $m, l \in \mathbb{R}$, if $a \in C^\infty(\mathbb{R}^d \times \mathbb{R}^d)$ and

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leq A_{\alpha\beta} \langle x \rangle^{l-|\beta|} \langle \xi \rangle^{m-|\alpha|}, \quad \alpha, \beta \in \mathbb{N}, x, \xi \in \mathbb{R}^d.$$

Recall that $\langle \xi \rangle := (1 + |\xi|^2)^{1/2}$.

We define $\Psi^{m,l}_{sc}(\mathbb{R}^d)$ accordingly.

Classical PsDOs

We can take a shortcut to define *classical* scattering pseudodifferential operators as follows.

Definition (classical scattering PsDOs on $\mathbb{R}^d$)

Let $\overline{\mathbb{R}^d}$ be the radial compactification of $\mathbb{R}^d$.

We define $S^{m,l}_{sc,cl}(\mathbb{R}^d \times \mathbb{R}^d) \subseteq S^{m,l}_{sc}(\mathbb{R}^d \times \mathbb{R}^d)$ as those a for which $a(x,\xi)\langle x\rangle^{-l}\langle\xi\rangle^{-d}$ extends to a smooth function $C^\infty(\overline{\mathbb{R}^d} \times \overline{\mathbb{R}^d})$.

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Definition (classical scattering PsDOs on $\mathbb{R}^d$)

Let $\overline{\mathbb{R}^d}$ be the radial compactification of $\mathbb{R}^d$.

We define $S^{m,l}_{sc,cl}(\mathbb{R}^d \times \mathbb{R}^d) \subseteq S^{m,l}_{sc}(\mathbb{R}^d \times \mathbb{R}^d)$ as those a for which $a(x,\xi)\langle x\rangle^{-l}\langle\xi\rangle^{-d}$ extends to a smooth function $C^\infty(\overline{\mathbb{R}^d} \times \overline{\mathbb{R}^d})$.

Accordingly, we define $\Psi^{m,l}_{sc,cl}(\mathbb{R}^d) \subseteq \Psi^{m,l}_{sc}(\mathbb{R}^d)$.

Classical PsDOs

We can take a shortcut to define *classical* scattering pseudodifferential operators as follows.

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Accordingly, we define $\Psi^{m,l}_{sc,cl}(\mathbb{R}^d) \subseteq \Psi^{m,l}_{sc}(\mathbb{R}^d)$.

Note that by Taylor's theorem, this is equivalent to $a(x,\xi)$ admitting asymptotic expansions of the right kind as $x \to \infty$, as $\xi \to \infty$, and as both $x,\xi \to \infty$.

Scattering cosphere bundle

Whereas on $\mathbb{R}^d$ we have $\Psi^m_{cl}(\mathbb{R}^d)/\Psi^{m-1}(\mathbb{R}^d) \simeq C^\infty(\mathbb{R}^d \times \mathbb{S}^{d-1})$ (also known as $C^\infty(S^*\mathbb{R}^d)$), for the scattering calculus the principal symbol is more complicated.

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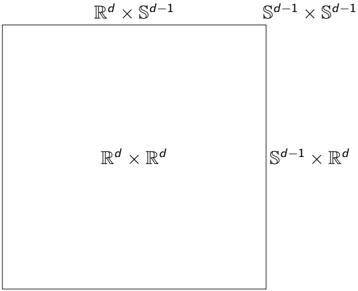
For $A \in \Psi^{m,l}_{sc,cl}(\mathbb{R}^d)$, the equivalence class $[A] \in \Psi^{m,l}(\mathbb{R}^d)/\Psi^{m-1,l-1}(\mathbb{R}^d)$ corresponds in a natural way to a smooth function in $C^\infty(\partial\overline{T^*\mathbb{R}^d})$, where

$$\partial\overline{T^*\mathbb{R}^d} := (\overline{\mathbb{R}^d} \times \overline{\mathbb{R}^d}) \setminus (\mathbb{R}^d \times \mathbb{R}^d) \simeq (\mathbb{R}^d \times \mathbb{S}^{d-1}) \sqcup (\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}) \sqcup (\mathbb{S}^{d-1} \times \mathbb{R}^d).$$



Examples

For $f \in C_c^\infty(\mathbb{R}^d)$, we have $M_f(1+\Delta)^{-\frac{d}{2}} \in \Psi_{sc,cl}^{-d,-\infty}(\mathbb{R}^d)$, we have $M_f(1+\Delta)^{-\frac{d}{2}} \in \mathcal{L}_{1,\infty}$, and

$$\mathrm{Tr}_\omega(M_f(1+\Delta)^{-\frac{d}{2}}) = \frac{1}{d(2\pi)^d} \int_{\partial \overline{T^*\mathbb{R}^d}} \sigma_{sc}^{-d,-d}(M_f(1+\Delta)^{-\frac{d}{2}})\, d\mu$$
$$\left(= \frac{\operatorname{vol} \mathbb{S}^{d-1}}{d(2\pi)^d} \int_{\mathbb{R}^d} f(x)\, dx \right).$$

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Now for the Fourier transform $f(\nabla)M_{\langle x\rangle^{-d}}$, we have $f(\nabla)M_{\langle x\rangle^{-d}} \in \Psi_{sc,cl}^{-\infty,-d}(\mathbb{R}^d)$, and

$$\mathrm{Tr}_\omega(f(\nabla)M_{\langle x\rangle^{-d}}) = \frac{1}{d(2\pi)^d}\int_{\partial\overline{T^*\mathbb{R}^d}} \sigma_{sc}^{-d,-d}(f(\nabla)M_{\langle x\rangle^{-d}})\,d\mu.$$

Known result

One might think that $\Psi^{-d,-d}_{sc,cl}(\mathbb{R}^d) \subseteq \mathcal{L}_{1,\infty}$, but this is not true:

$$M^{-d}_{\langle x\rangle}(1+\Delta)^{-\frac{d}{2}} \notin \mathcal{L}_{1,\infty}.$$

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Theorem (Nicola 2003)

Let $P \in \Psi^{-d,-d-1}_{sc,cl}(\mathbb{R}^d)$. Then $P \in \mathcal{L}_{1,\infty}$, and

$$\mathrm{Tr}_\omega(P) = \frac{1}{d(2\pi)^d}\int_{\mathbb{R}^d\times\mathbb{S}^{d-1}} \sigma^{-d,-d}_{sc}(P)\,d\mu.$$

If $P \in \Psi^{-d-1,-d}_{sc,cl}(\mathbb{R}^d)$, the same formula holds with integral over $\mathbb{S}^{d-1}\times\mathbb{R}^d$.

If $P \in \Psi^{-d,-d}_{sc,cl}(\mathbb{R}^d)$, then

$$\lim_{N\to\infty}\frac{1}{\big(\log(N+2)\big)^2}\sum_{n=0}^{N}\lambda(n,P) = \int_{\mathbb{S}^{d-1}\times\mathbb{S}^{d-1}} \sigma^{-d,-d}_{sc}(P)\,d\mu.$$

New result

Theorem (H.–Levitina–McDonald–Sukochev–Zanin, WIP)

Let $P \in \Psi_{sc,cl}^{-d,-d}(\mathbb{R}^d)$. Then $P \in \mathcal{L}_{1,\infty}$ if and only if $\sigma_{sc}^{-d,-d}(P) \in L_1(\partial\overline{T^*\mathbb{R}^d})$, in which case

$$\mathrm{Tr}_\omega(P) = \frac{1}{d(2\pi)^d} \int_{\partial\overline{T^*\mathbb{R}^d}} \sigma_{sc}^{-d,-d}(P)\, d\mu.$$

Next step: scattering metrics?

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Next step: scattering metrics?

Note: if $P \in \mathcal{L}_{1,\infty} \cap \Psi^{-d,-d}_{sc,cl}(\mathbb{R}^d)$, then its principal symbol is zero at the corner $\mathbb{S}^{d-1} \times \mathbb{S}^{d-1}$. The relevant part of the measure $d\mu$ here is the Lebesgue measure on $\mathbb{R}^d \times \mathbb{S}^{d-1} \sqcup \mathbb{S}^{d-1} \times \mathbb{R}^d$.

Thanks

Thanks for the invite!