The Joys of MOIs

University of Bonn
Oberseminar Global Analysis and Operator Algebras

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Summary of this talk

1. Motivation for studying MOIs
2. Pseudodifferential calculus
3. MOIs of pseudodifferential operators

This talk is based on joint work with Edward McDonald and Teun van Nuland.

Part 1: Motivation

Exhibit A

We use the Chern character of $(\mathcal{A}, \mathcal{H}, D)$ in entire cyclic cohomology (cf. [2]) given in the most efficient manner by the JLO formula, which defines the components of an entire cocycle in the (b, B) bicomplex:

$$(90) \qquad \psi_n(a^0, \ldots, a^n) = \sqrt{2i} \int_{\sum_0^n v_i = 1, v_i \geq 0}$$

$$\operatorname{Trace}\left(a^0\, e^{-v_0 D^2}\, [D, a^1]\, e^{-v_1 D^2} \ldots e^{-v_{n-1} D^2}\, [D, a^n]\, e^{-v_n D^2}\right), \qquad \forall a^j \in \mathcal{A}$$

where n is odd.

We introduce a parameter ϵ by replacing D^2 by ϵD^2, which yields a cocycle ψ_n^ϵ which is cohomologous to ψ_n. One has moreover

$$(91) \qquad \psi_n^\epsilon(a^0, \ldots, a^n) = \sqrt{2i} \left(\int_{\sum_0^n v_i = 1} \theta(\epsilon\, v_0, \ldots, \epsilon\, v_n)\, \pi\, dv_i \right) \epsilon^{n/2},$$

From [ConnesMoscovici1995]

Exhibit B

Let us now show that if $b \in \cap$ Dom $L^k R^q$ then $b \in$ Dom δ. The proof is more subtle than one would expect, because the obvious argument, using

$$|D| = \pi^{-1} \int_0^\infty \frac{D^2}{D^2+\mu} \, \mu^{-1/2} \, d\mu \, ,$$

requires some care. Indeed, one gets from the above

$$[|D|, b] = \pi^{-1} \int_0^\infty (D^2+\mu)^{-1} \, [D^2, b] \, (D^2+\mu)^{-1} \, \mu^{1/2} \, d\mu \, .$$

We can replace $[D^2, b]$ by $|D|$, which has the same size, and get

$$\int_0^\infty (D^2+\mu)^{-2} \, |D| \, \mu^{1/2} \, d\mu = \int_0^\infty (1+t)^{-2} \, t^{1/2} \, dt \, .$$

For this to work, we need to move $[D^2, b]$ in front of the above integral, i.e. use the finiteness of the norm of

$$\int_0^\infty \underbrace{[(D^2+\mu)^{-1}, [D^2, b]]}_{-(D^2+\mu)^{-1}[D^2,[D^2,b]](D^2+\mu)^{-1}} \, (D^2+\mu)^{-1} \, \mu^{1/2} \, d\mu \, .$$

This finiteness follows from:

1) $(D^2+\mu)^{-1} \, [D^2, [D^2, b]]$ bounded since $b \in \mathrm{Dom}\, L^2$

2) $\int_0^\infty \|(D^2+\mu)^{-2}\| \, \mu^{1/2} \, d\mu \leq C \int_0^1 \mu^{1/2} \, d\mu + \int_1^\infty \mu^{-3/2} \, d\mu < \infty$.

Once $[D^2, b]$ is moved in front the above calculation applies.

From [ConnesMoscovici1995]

Exhibit C

Now, onwards with the computation, the first part of which is straightforward:

$$\begin{aligned}[\Delta^{-z}, A]B &= \frac{1}{2\pi i}\int \lambda^{-z}[(\lambda-\Delta)^{-1}, A]B \, d\lambda \\ &= \frac{1}{2\pi i}\int \lambda^{-z}(\lambda-\Delta)^{-1}[\Delta, A](\lambda-\Delta)^{-1}B \, d\lambda \\ &= \int \lambda^{-z}(\lambda-\Delta)^{-1}[\Delta, A]B(\lambda-\Delta)^{-1} \, d\lambda \\ &\quad + \int \lambda^{-z}(\lambda-\Delta)^{-1}[\Delta, A](\lambda-\Delta)^{-1}[\Delta, B](\lambda-\Delta)^{-1} \, d\lambda.\end{aligned}$$

(In the last step we did two things at once: we commuted B past $(\lambda-\Delta)^{-1}$ and we then used the formula $[S^{-1}, T] = S^{-1}[T, S]S^{-1}$.) The operators $[\Delta, A]$ and $[\Delta, B]$ have orders 1 and 2, respectively.

Before going on, we shall introduce some better notation for our contour integrals.

2.5 Definition. If $D_0, \ldots, D_p$ are differential operators on the closed manifold M, then denote by $I_z(D_0, \ldots, D_p)$ the integral

$$\frac{1}{2\pi i}\int \lambda^{-z} D_0(\lambda-\Delta)^{-1} \cdots D_p(\lambda-\Delta)^{-1} \, d\lambda$$

(in the integral, copies of $(\lambda-\Delta)^{-1}$ alternate with the operators D_j). The integral converges if $\mathrm{Re}(z) < n$, in the sense we discussed above, and defines an operator on $C^\infty(M)$.

From [Higson2003]

Exhibit D

Theorem 4.2 (Semifinite Odd Local Index Theorem). *Let $(\mathcal{A}, \mathcal{H}, \mathcal{D})$ be an odd finitely summable QC^∞ spectral triple with spectral dimension $p \geq 1$. Let $N = [p/2] + 1$ where $[\cdot]$ denotes the integer part, and let $u \in \mathcal{A}$ be unitary. Then*

1) $$sf(\mathcal{D}, u^*\mathcal{D}u) = \frac{1}{\sqrt{2\pi i}} res_{r=(1-p)/2} \left(\sum_{m=1,odd}^{2N-1} \phi_m^r(Ch_m(u)) \right)$$

where for $a_0, \dots, a_m \in \mathcal{A}$, $l = \{a + iv : v \in \mathbf{R}\}$, $0 < a < 1/2$, $R_s(\lambda) = (\lambda - (1 + s^2 + \mathcal{D}^2))^{-1}$ and $r > 0$ we define $\phi_m^r(a_0, a_1, \dots, a_m)$ to be

$$\frac{-2\sqrt{2\pi i}}{\Gamma((m+1)/2)} \int_0^\infty s^m \tau \left(\frac{1}{2\pi i} \int_l \lambda^{-p/2-r} a_0 R_s(\lambda)[\mathcal{D}, a_1] R_s(\lambda) \cdots [\mathcal{D}, a_m] R_s(\lambda) d\lambda \right) ds.$$

In particular the sum on the right hand side of 1) *analytically continues to a deleted neighbourhood of $r = (1-p)/2$ with* at worst *a simple pole at $r = (1-p)/2$. Moreover, the complex function-valued cochain $(\phi_m^r)_{m=1,odd}^{2N-1}$ is a (b, B) cocycle for $\mathcal{A}$ modulo functions holomorphic in a half-plane containing $r = (1-p)/2$.*

From [CareyPhillipsRennieSukochev2006]

Exhibit E

Let us introduce the following convenient notation (cf. [10]). If $A_0, \ldots, A_n$ are operators, we define a t-dependent quantity by

$$\langle A_0, \ldots, A_n \rangle_n := t^n \operatorname{Tr} \int_{\Delta_n} A_0 e^{-s_0 t D^2} A_1 e^{-s_1 t D^2} \cdots A_n e^{-s_n t D^2} \, d^n s. \tag{3}$$

Note the difference in notation with [10], for which the same symbol is used for the supertrace of the same expression, rather than the trace. Also, we are integrating over the 'inflated' n-simplex $t\Delta^n$, yielding the factor t^n. The forms $\langle A_0, \ldots, A_n \rangle$ satisfy, *mutatis mutandis*, the following properties.

Lemma 7. *(See [10].) In each of the following cases, we assume that the operators A_i are such that each term is well defined*:

1. $\langle A_0, \ldots, A_n \rangle_n = \langle A_i, \ldots, A_n, \ldots, A_{i-1} \rangle_n$;
2. $\langle A_0, \ldots, A_n \rangle_n = \sum_{i=0}^n \langle 1, \ldots, A_i, \ldots, A_n, A_0, \ldots, A_{i-1} \rangle_n$;
3. $\sum_{i=0}^n \langle A_0, \ldots, [D, A_i], \ldots, A_n \rangle_n = 0$;
4. $\langle A_0, \ldots, [D^2, A_i], \ldots, A_n \rangle_n = \langle A_0, \ldots, A_{i-1} A_i, , \ldots, A_n \rangle_{n-1} - \langle A_0, \ldots, A_i A_{i+1}, \ldots, A_n \rangle_{n-1}$.

From [vanSuijlekom2011]

Multiple operator integrals

Let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ be such that

$$\phi(\lambda_0, \ldots, \lambda_n) = \int_\Omega a_0(\lambda_0, \omega) \cdots a_n(\lambda_n, \omega) d\nu(\omega),$$

with finite measure space (Ω, ν) and measurable and bounded $a_j : \mathbb{R} \times \Omega \to \mathbb{C}$.

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with finite measure space (Ω, ν) and measurable and bounded $a_j : \mathbb{R} \times \Omega \to \mathbb{C}$.

Let $H_0, \dots, H_n$ be self-adjoint, for $V_1, \dots, V_n \in B(\mathcal{H})$ define the MOI

$$T_\phi^{H_0, \dots, H_n}(V_1, \dots, V_n)\psi$$
$$:= \int_\Omega a_0(H_0, \omega) V_1 a_1(H_1, \omega) \cdots V_n a_n(H_n, \omega) \psi d\nu(\omega), \quad \psi \in \mathcal{H}.$$

Then,

$$T_\phi^{H_0, \dots, H_n} : B(\mathcal{H}) \times \cdots \times B(\mathcal{H}) \to B(\mathcal{H})$$

and this does not depend on how we represent ϕ (its *symbol*).

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For a large enough contour γ,

$$\begin{aligned} f(A) - f(B) &= \frac{1}{2\pi i} \int_\gamma f(z)(z-A)^{-1}\,dz - \frac{1}{2\pi i} \int_\gamma f(z)(z-B)^{-1}\,dz \\ &= \frac{1}{2\pi i} \int_\gamma f(z)(z-A)^{-1}(A-B)(z-B)^{-1}\,dz \\ &= T^{A,B}_{f^{[1]}}(A-B). \end{aligned}$$

The symbol $f^{[1]}$ here is very typical.

Divided differences

Symbols of MOIs encountered in the wild are almost always divided differences, which are defined recursively for $f \in C^n(\mathbb{R})$ as

$$f^{[0]}(\lambda) := f(\lambda);$$
$$f^{[n]}(\lambda_0, \dots, \lambda_n) := \frac{f^{[n-1]}(\lambda_0, \dots, \lambda_{n-1}) - f^{[n-1]}(\lambda_1, \dots, \lambda_n)}{\lambda_0 - \lambda_n},$$

with an appropriate limit if $\lambda_0 = \lambda_n$. In particular,

$$\frac{1}{n!} f^{(n)}(\lambda) = f^{[n]}(\lambda, \dots, \lambda).$$

Example MOIs 2

For example, the JLO cocycle is

$$\int_{\Delta_n} \mathrm{Tr}\big(\eta a_0 e^{-t_0 D^2}[D, a_1]e^{-t_1 D^2} \cdots [D, a_n]e^{-t_n D^2}\big)\, dt$$
$$= \mathrm{Tr}(\eta a_0 T^{D^2}_{f^{[n]}}([D, a_1], \ldots, [D, a_n])),$$

with $f(x) = \exp(-x)$.

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- $\frac{d^n}{dt^n} f(H+tV)|_{t=0} = T^{H,\ldots,H}_{f^{[n]}}(V, \ldots, V)$,

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Furthermore, MOIs can systematise operator integral techniques in NCG.

A problem

If you write, like in *The Local Index Formula in Noncommutative Geometry* by Nigel Higson, for a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and $a \in \mathcal{A}$,

$$
\begin{aligned}
[D^{-2z}, a] &= \left[\int_\gamma \lambda^{-z}(\lambda - D^2)^{-1} d\lambda, a \right] \\
&= \int_\gamma \lambda^{-z} [(\lambda - D^2)^{-1}, a] d\lambda \\
&= \int_\gamma \lambda^{-z}(\lambda - D^2)^{-1} [D^2, a] (\lambda - D^2)^{-1} d\lambda,
\end{aligned}
$$

then $[D^2, a] \notin B(\mathcal{H})$, so this is not a standard MOI.

Part 2: Abstract pseudodifferential calculus in the style of Connes–Moscovici, Higson, Guillemin

Pseudodifferential operators

On $\mathbb{R}^d$, a differential operator $L = \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha$ can be written as

$$L = \mathcal{F}^{-1} \circ M_{p_L} \circ \mathcal{F},$$

where M_{p_L} indicates multiplying with the polynomial $p_L(x,\xi) := \sum_{|\alpha|\leq k} a_\alpha(x)\xi^\alpha$.

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Generally speaking, a pseudodifferential operator of order k on $\mathbb{R}^d$ is an operator of the form $L = \mathcal{F}^{-1} \circ M_{p_L} \circ \mathcal{F}$ where the function p_L is more general, such that

$$L : \mathcal{H}^{s+k,2} \to \mathcal{H}^{s,2},$$

where

$$\mathcal{H}^{s,2}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}\big[(1+|\xi|^2)^{s/2}\mathcal{F}f\big] \in L_2(\mathbb{R}^n)\},$$

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On a smooth manifold M, we can define classes of pseudodifferential operators on $L_2(M)$ by patching together operators like this.

Sobolev spaces

Given an invertible, positive self-adjoint operator Θ on a separable Hilbert space $\mathcal{H}$, we can define the 'Sobolev' spaces $\mathcal{H}^s$, $s \in \mathbb{R}$, as the completion of $\operatorname{dom} \Theta^s$ under the norm

$$\|\xi\|_s^2 = \langle \xi, \xi \rangle_s := \langle \Theta^s \xi, \Theta^s \xi \rangle_{\mathcal{H}} = \|\Theta^s \xi\|^2, \quad \xi \in \operatorname{dom} \Theta^s.$$

This forms a Hilbert space. We have continuous embeddings

$$\mathcal{H}^t \subseteq \mathcal{H}^s, \quad s \leq t,$$

because

$$\|\Theta^s \xi\| \leq \|\Theta^{s-t}\|_\infty \|\Theta^t \xi\|.$$

We put

$$\mathcal{H}^\infty := \bigcap_{s \in \mathbb{R}} \mathcal{H}^s, \quad \mathcal{H}^{-\infty} := \bigcup_{s \in \mathbb{R}} \mathcal{H}^s,$$

and we get for free that $\mathcal{H}^\infty$ is dense in $\mathcal{H}$.

Analytic order

Even though Θ itself is an unbounded operator on $\mathcal{H}$, if we regard it as an operator

$$\Theta : \mathcal{H}^1 \to \mathcal{H}^0 = \mathcal{H},$$

it is a perfectly good bounded operator:

$$\|\Theta\|_{\mathcal{H}^1 \to \mathcal{H}^0} = \sup_{\xi : \|\Theta\xi\| \leq 1} \|\Theta\xi\| = 1.$$

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We define $\mathrm{op}^r(\Theta)$ for $r \in \mathbb{R}$ as those $T : \mathcal{H}^\infty \to \mathcal{H}^\infty$ that extend to a bounded operator

$$T : \mathcal{H}^{s+r} \to \mathcal{H}^s, \quad s \in \mathbb{R}.$$

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We define $\mathrm{OP}^r(\Theta)$ as those $T \in \mathrm{op}^r(\Theta)$ for which $\delta_\Theta^n(T) \in \mathrm{op}^r(\Theta)$ for all $n \in \mathbb{N}$, where $\delta_\Theta(T) = [\Theta, T]$.

Examples

- If Δ is the Laplace operator on $\mathbb{R}^n$, setting $\Theta = (1-\Delta)^{1/2}$ gives the standard (Bessel potential) Sobolev spaces. The k-th order (pseudo)differential operators are contained in $\mathrm{OP}^k(\Theta)$.

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- Taking $\Theta = (1-\Delta)^{1/2}$ where Δ is the sub-Laplacian on a stratified Lie group gives the Sobolev spaces defined by Folland and Stein.
- For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ it makes sense to put $\Theta = (1+D^2)^{1/2}$. Then for example $D \in \mathrm{OP}^1(\Theta)$, and for a *regular* spectral triple $a, [D, a] \in \mathrm{OP}^0(\Theta)$ for all $a \in \mathcal{A}$.

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- If Θ is bounded, $\mathcal{H}^s \simeq \mathcal{H}$ and $\mathrm{op}^r(\Theta) = B(\mathcal{H})$ for all $s, r \in \mathbb{R}$.

Elliptic operators

Our goal is to construct MOIs where all operators are in $\mathrm{op}(\Theta)$. First, we need a functional calculus for such operators. Analogously to usual notions of pseudodifferential operators, a functional calculus can be constructed for *elliptic* operators.

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We define $T \in \mathrm{op}^r(\Theta)$ to be Θ-*elliptic*, if there is a parametrix $P \in \mathrm{op}^{-r}(\Theta)$ such that

$$TP = 1_{\mathcal{H}^\infty} + \mathrm{op}^{-\infty}(\Theta);$$
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By a Borel Lemma argument, it suffices if

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Elliptic operators 2

For any spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and $\Theta = (1 + D^2)^{1/2}$, we have that $D \in \mathrm{op}^1(\Theta)$ is Θ-elliptic. Furthermore, $D + V$ is Θ-elliptic if $V \in \mathrm{op}^r(\Theta)$ with $r < 1$.

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If $T \in \mathrm{op}^r(\Theta)$ is Θ-elliptic,

- If $x \in \mathcal{H}^{-\infty}$, then $Tx \in \mathcal{H}^s$ implies that $x \in \mathcal{H}^{s+r}$ (*elliptic regularity*).
- If $T : \mathcal{H}^r \subseteq \mathcal{H}^0 \to \mathcal{H}^0$ (i.e. $r \geq 0$) is a symmetric operator, then it is self-adjoint given domain $\mathcal{H}^r$. This situation will be referred to as 'T is Θ-elliptic and symmetric'.

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$$T : \mathcal{H}^{r+s} \subseteq \mathcal{H}^s \to \mathcal{H}^s$$

is self-adjoint for any other $s \in \mathbb{R}$. In fact, these operators need not even be symmetric or normal.

Functional calculus

We write $f \in L_\infty^\beta(\mathbb{R})$ for some $\beta \in \mathbb{R}$ if $f(x)(1+x^2)^{-\beta/2} \in L_\infty(\mathbb{R})$.

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H.–McDonald–van Nuland (2024)

Let $T \in \mathrm{op}^r(\Theta)$, $r > 0$, be Θ-elliptic and symmetric. If $f \in L_\infty^\beta(\mathbb{R})$, $\beta \in \mathbb{R}$, then

$$f(T) \in \mathrm{op}^{r\beta}(\Theta).$$

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We write $f \in L^{\beta}_{\infty}(\mathbb{R})$ for some $\beta \in \mathbb{R}$ if $f(x)(1+x^2)^{-\beta/2} \in L_{\infty}(\mathbb{R})$.

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$$f(T) \in \mathrm{op}^{r\beta}(\Theta).$$

Furthermore, if A is self-adjoint on $\mathcal{H}$, $A \in \mathrm{op}^t(\Theta)$, $t \in \mathbb{R}$, and A commutes strongly with T, then for $f \in L^{\beta}_{\infty}(\mathbb{R})$, $\beta \geq 0$, we have

$$f(A) \in \mathrm{op}^{t\beta}(\Theta).$$

This second part applies for example to $i\frac{d}{dx}$ in $\mathrm{op}(1-\Delta)^{1/2}$ on $\mathbb{R}^d$.

Part 3: MOIs as pseudodifferential operators

Unbounded MOIs

H.–McDonald–van Nuland (2024)

Let $H_i \in \mathrm{op}^{h_i}(\Theta)$, $h_i > 0$ Θ-elliptic and symmetric for $i = 0, \ldots, n$, and $X_i \in \mathrm{op}^{r_i}(\Theta)$ for $i = 1, \ldots, n$. Let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ such that

$$\phi(\lambda_0, \ldots, \lambda_n) = \int_\Omega a_0(\lambda_0, \omega) \cdots a_n(\lambda_n, \omega) d\nu(\omega),$$

with finite measure space (Ω, ν) with $a_j(x, \omega)(1 + x^2)^{-\beta_j/2} : \mathbb{R} \times \Omega \to \mathbb{C}$ measurable and bounded. Then for $\psi \in \mathcal{H}^\infty$,

$$T_\phi^{H_0,\ldots,H_n}(X_1, \ldots, X_n)\psi := \int_\Omega a_0(H_0, \omega) X_1 a_1(H_1, \omega) \cdots X_n a_n(H_n, \omega) \psi d\nu(\omega)$$

is a well-defined vector in $\mathcal{H}^\infty$ independent of the representation of ϕ, and

$$T_\phi^{H_0,\ldots,H_n} : \mathrm{op}^{r_1}(\Theta) \times \cdots \times \mathrm{op}^{r_n}(\Theta) \to \mathrm{op}^{\sum_j r_j + \sum_j \beta_j h_j}(\Theta).$$

Unbounded MOIs: the useful bit

If $f \in C^{n+2}(\mathbb{R})$, and $f^{(k)} \in L_\infty^{\beta-k}(\mathbb{R})$ for $k = 0, \dots, n+2$, then for $H \in \mathrm{op}^h(\Theta)$, $h > 0$ Θ-elliptic and symmetric, and $X_i \in \mathrm{op}^{r_i}(\Theta)$,

$$T_{f^{[n]}}^{H,\dots,H}(X_1, \dots, X_n) \in \bigcap_{\varepsilon>0} \mathrm{op}^{(\beta-n)h+\sum_j r_j+\varepsilon}(\Theta).$$

If $f \in C^\infty(\mathbb{R})$ and $f^{(k)} \in L_\infty^{\beta-k}(\mathbb{R})$ for all $k \in \mathbb{N}$, we write $f \in S^\beta(\mathbb{R})$.

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For integral we saw earlier, with the appropriate conditions on D and a, we have

$$\int_\gamma \lambda^{-z}(\lambda - D^2)^{-1}[D^2, a](D^2 - \lambda)^{-1} d\lambda = T_{f^{[1]}}^{D^2,D^2}([D^2, a]) \in \bigcap_{\varepsilon>0} \mathrm{op}^{-2\Re(z)-1+\varepsilon}(\Theta),$$

with $f(x) = x^{-z}$ and $\Theta = (1 + D^2)^{\frac{1}{2}}$.

Two rules

MOIs as we defined them come with two identities:

1. $f(A) - f(B) = T^{A,B}_{f^{[1]}}(A - B)$;
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and the higher order analogues (since $T^{H}_{f^{[0]}}() = f(H)$)

1. $T^{H_0,\ldots,A,\ldots,H_n}_{f^{[n]}}(V_1, \ldots, V_n) - T^{H_0,\ldots,B,\ldots,H_n}_{f^{[n]}}(V_1, \ldots, V_n)$
$= T^{H_0,\ldots,A,B,\ldots,H_n}_{f^{[n+1]}}(V_1, \ldots, A - B, \ldots, V_n)$;

2. $T^{H_0,\ldots,H_n}_{f^{[n]}}(V_1, \ldots, V_{j-1}, aV_j, \ldots, V_n) - T^{H_0,\ldots,H_n}_{f^{[n]}}(V_1, \ldots, V_{j-1}a, V_j, \ldots, V_n)$
$= T^{H_0,\ldots,H_j,H_j,\ldots,H_n}_{f^{[n+1]}}(V_1, \ldots, V_{j-1}, [H_j, a], V_{j+1}, \ldots, V_n)$.

Taylor expansion

The first rule on its own gives a Taylor expansion:

$$\begin{aligned} f(H+V) &\overset{(1)}{=} f(H) + T_{f^{[1]}}^{H+V,H}(V) \\ &\overset{(1)}{=} f(H) + T_{f^{[1]}}^{H,H}(V) + T_{f^{[2]}}^{H+V,H,H}(V,V), \end{aligned}$$

and repeat. We get for all $N \in \mathbb{N}$

$$f(H+V) = \sum_{n=0}^{N} T_{f^{[n]}}^{H,\ldots,H}(V,\ldots,V) + T_{f^{[N+1]}}^{H+V,H,\ldots,H}(V,\ldots,V).$$

Note: if H and V commute,

$$T_{f^{[n]}}^{H,\ldots,H}(V,\ldots,V) = \frac{1}{n!} f^{(n)}(H) V^n.$$

Commutator expansion

In similar manner, by rule (2) we get

$$T^{H,\dots,H}_{f^{[n]}}(V_1,\dots,V_n) = V_1\, T^{H,\dots,H}_{f^{[n]}}(1, V_2,\dots,V_n) + T^{H,\dots,H}_{f^{[n+1]}}([H,V],1,V_2,\dots,V_n),$$

repeating and remembering that $T^{H,\dots,H}_{f^{[n]}}(1,\dots,1) = \frac{1}{n!} f^{(n)}(H)$, we get

$$T^{H,\dots,H}_{f^{[n]}}(V_1,\dots,V_n) = \sum_{m=0}^{N} \sum_{m_1+\dots+m_n=m} \frac{C_{m_1,\dots,m_n}}{(n+m)!} \delta_H^{m_1}(X_1)\cdots\delta_H^{m_n}(X_n) f^{(n+m)}(H) + S^N_{H,V}.$$

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The combinatorics to get this expression is exactly the same as how one gets the cocycle of the local index formula, writing $A^{(k)} := \delta^n_{D^2}(A)$,

$$\begin{aligned} &\phi_n(a_0,\dots,a_n) \\ &= \sum_{|k|,q\geq 0} c_{n,k,q}\operatorname{Res}_{z=0} z^q \operatorname{Tr}\left(a_0[D,a_1]^{(k_1)}\cdots[D,a_n]^{(k_n)}|D|^{-2|k|-2z-n}\right), \end{aligned}$$

Asymptotic expansions

We say that $T \sim \sum_{k=0}^{\infty} T_k$ for $T, T_k \in \mathrm{op}(\Theta)$ if

$$T - \sum_{k=1}^{N} T_k \in \mathrm{op}^{m_N}(\Theta), \quad m_N \downarrow -\infty.$$

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If $f \in S^{\beta}(\mathbb{R})$, if $H \in \mathrm{op}^h(\Theta)$, $h > 0$ is Θ-elliptic and symmetric, and if $V \in \mathrm{op}^r(\Theta)$ with $r < h$, then

$$f(H+V) \sim \sum_{n=0}^{\infty} T_{f^{[n]}}^{H,\ldots,H}(V,\ldots,V).$$

If furthermore $\delta_H^n(V) \in \mathrm{op}^{r+n(h-\varepsilon)}$ for some $\varepsilon > 0$ (for example $H = \Theta$, $V \in \mathrm{OP}^r$)

$$T_{f^{[n]}}^{H,\ldots,H}(V,\ldots,V) \sim \sum_{m=0}^{\infty} \sum_{m_1+\cdots+m_n=m} \frac{C_{m_1,\ldots,m_n}}{(n+m)!} \delta_H^{m_1}(V) \cdots \delta_H^{m_n}(V) f^{(n+m)}(H).$$

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Combined,

$$f(H+V) \sim \sum_{n,m=0}^{\infty} \sum_{m_1+\cdots+m_n=m} \frac{C_{m_1,\ldots,m_n}}{(n+m)!} \delta_H^{m_1}(V)\cdots\delta_H^{m_n}(V) f^{(n+m)}(H).$$

A familiar expansion

Recall that

$$[f(\Theta), X] = T^{\Theta,\Theta}_{f^{[1]}}([\Theta, X]).$$

Therefore, for $X \in \mathrm{OP}^r(\Theta)$, the expansions on the last slide give

$$[f(\Theta), X] \sim \sum_{k=1}^{\infty} \frac{1}{k!} \delta_{\Theta}^{k}(X) f^{(k)}(\Theta).$$

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In particular,

$$[\Theta^\alpha, X] \sim \sum_{k=1}^{\infty} \binom{\alpha}{k} \delta_\Theta^k(X) \Theta^{\alpha-k}, \quad \alpha \in \mathbb{C},$$

and

$$[\log(\Theta), X] \sim \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{k} \delta_\Theta^k(X) \Theta^{-k},$$

and we have that $[\Theta^\alpha, X] \in \mathrm{OP}^{r+\Re(\alpha)-1}(\Theta)$ and $[\log(\Theta), X] \in \mathrm{OP}^{r-1}(\Theta)$.

Asymptotic trace expansions

H.–McDonald–van Nuland (2024)

Let $(\mathcal{A}, \mathcal{H}, D)$ be a regular s-summable spectral triple $((1+D^2)^{-\frac{1}{2}} \in \mathcal{L}_s)$. Let V self-adjoint and bounded, generated by $\mathcal{A}$ and D. Then if f has sufficient regularity, as $t \to 0$,

$$\begin{aligned}
&\mathrm{Tr}(f(tD+tV)) \\
&= \sum_{n=0}^{N} \sum_{m=0}^{N} \sum_{m_1+\cdots+m_n=m} t^{n+m} \frac{C_{m_1,\ldots,m_n}}{(n+m)!} \mathrm{Tr}\big(\delta_D^{m_1}(V)\cdots\delta_D^{m_n}(V) f^{(n+m)}(tD)\big) \\
&\quad + O(t^{N+1-s}).
\end{aligned}$$

Bonus: Functional calculus for OP

If $A \in \mathrm{op}^r(\Theta)$ is Θ-elliptic and symmetric, then by rule (2) we know that for $f \in L_\infty^\beta(\mathbb{R})$

$$f(A) \in \mathrm{op}^{\beta r}(\Theta),$$

$$[\Theta, f(A)] = T^{A,A}_{f^{[1]}}([\Theta, A]),$$

and similar expressions hold for $\delta_\Theta^n(f(A))$. If $A \in \mathrm{OP}^r(\Theta)$, then we can deduce what the order is of these expressions if $f \in S^\beta(\mathbb{R})$, so that

$$\delta_\Theta^n(f(A)) \in \bigcap_{\varepsilon>0} \mathrm{op}^{r\beta+\varepsilon}(\Theta).$$

We therefore conclude that $f(A) \in \bigcap_{\varepsilon>0} \mathrm{OP}^{r\beta+\varepsilon}(\Theta)$.

Thanks

Thank you for your attention!