Unbounded operator integrals and Quantum Field Theory

University of Potsdam Analysis Seminar

Eva-Maria Hekkelman

Max Planck Institute for Mathematics Bonn

May 20, 2026

Summary of this talk

1. Motivation
2. Pseudodifferential calculus
3. MOIs of pseudodifferential operators
4. Applications in QFT

This talk is based on joint work with Edward McDonald, Teun van Nuland, and Jesse Reimann.

Part 1: Motivation

Fundamental question

If you have two operators H, V on a Hilbert space and a suitable function $f : \mathbb{C} \to \mathbb{C}$, you might come across one of the following objects:

- $f(H+V) - f(H)$;

Fundamental question

If you have two operators H, V on a Hilbert space and a suitable function $f : \mathbb{C} \to \mathbb{C}$, you might come across one of the following objects:

- $f(H+V) - f(H)$;
- $[f(H), V]$;

Fundamental question

If you have two operators H, V on a Hilbert space and a suitable function $f : \mathbb{C} \to \mathbb{C}$, you might come across one of the following objects:

- $f(H+V) - f(H)$;
- $[f(H), V]$;
- $\frac{d}{dt} f(H+tV)|_{t=0}$.

Fundamental question

If you have two operators H, V on a Hilbert space and a suitable function $f : \mathbb{C} \to \mathbb{C}$, you might come across one of the following objects:

- $f(H+V) - f(H)$;
- $[f(H), V]$;
- $\frac{d}{dt} f(H+tV)|_{t=0}$.

But how to study the properties of these operators?

Computation

For example, let $A, B \in B(\mathcal{H})$ and $f : \mathbb{C} \to \mathbb{C}$ holomorphic. For a large enough contour γ,

Computation

For example, let $A, B \in B(\mathcal{H})$ and $f : \mathbb{C} \to \mathbb{C}$ holomorphic. For a large enough contour γ,

$$\begin{aligned}
\frac{d}{dt} f(H+tV)\Big|_{t=0} &:= \lim_{t\to 0} \frac{f(H+tV) - f(H)}{t} \\
&= \lim_{t\to 0} \frac{1}{t}\bigg(\frac{1}{2\pi i} \int_\gamma f(z)(z-H-tV)^{-1}\, dz \\
&\qquad - \frac{1}{2\pi i} \int_\gamma f(z)(z-H)^{-1}\, dz \bigg) \\
&= \lim_{t\to 0} \frac{1}{2\pi i} \int_\gamma f(z)(z-H-tV)^{-1} V (z-H)^{-1}\, dz \\
&= \frac{1}{2\pi i} \int_\gamma f(z)(z-H)^{-1} V (z-H)^{-1}\, dz.
\end{aligned}$$

Computation 2

Similarly,

$$\frac{d^n}{dt^n} f(H+tV)\Big|_{t=0} = \frac{n!}{2\pi i}\int_\gamma f(z)(z-H)^{-1}V(z-H)^{-1}\cdots V(z-H)^{-1}\,dz,$$

$$f(H+V)-f(H) = \frac{1}{2\pi i}\int_\gamma f(z)(x-H-V)^{-1}V(z-H)^{-1}\,dz,$$

$$[f(H),V] = \frac{1}{2\pi i}\int_\gamma f(z)(x-H)^{-1}[H,V](z-H)^{-1}\,dz.$$

Multiple operator integrals

Let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ be such that

$$\phi(\lambda_0, \dots, \lambda_n) = \int_\Omega a_0(\lambda_0, \omega) \cdots a_n(\lambda_n, \omega) d\nu(\omega),$$

with finite measure space (Ω, ν) and measurable and bounded $a_j : \mathbb{R} \times \Omega \to \mathbb{C}$.

Multiple operator integrals

Let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ be such that

$$\phi(\lambda_0, \ldots, \lambda_n) = \int_\Omega a_0(\lambda_0, \omega) \cdots a_n(\lambda_n, \omega) d\nu(\omega),$$

with finite measure space (Ω, ν) and measurable and bounded $a_j : \mathbb{R} \times \Omega \to \mathbb{C}$.

Let $H_0, \ldots, H_n$ be self-adjoint, for $V_1, \ldots, V_n \in B(\mathcal{H})$ define the MOI

$$T_\phi^{H_0,\ldots,H_n}(V_1, \ldots, V_n)\psi$$
$$:= \int_\Omega a_0(H_0, \omega) V_1 a_1(H_1, \omega) \cdots V_n a_n(H_n, \omega) \psi d\nu(\omega), \quad \psi \in \mathcal{H}.$$

Then,

$$T_\phi^{H_0,\ldots,H_n} : B(\mathcal{H}) \times \cdots \times B(\mathcal{H}) \to B(\mathcal{H})$$

and this does not depend on how we represent ϕ (its *symbol*).

Divided differences

Symbols of MOIs encountered in the wild are almost always divided differences, which are defined recursively for $f \in C^n(\mathbb{R})$ as

$$f^{[0]}(\lambda) := f(\lambda);$$
$$f^{[n]}(\lambda_0, \ldots, \lambda_n) := \frac{f^{[n-1]}(\lambda_0, \ldots, \lambda_{n-1}) - f^{[n-1]}(\lambda_1, \ldots, \lambda_n)}{\lambda_0 - \lambda_n},$$

with an appropriate limit if $\lambda_0 = \lambda_n$. In particular,

$$\frac{1}{n!} f^{(n)}(\lambda) = f^{[n]}(\lambda, \ldots, \lambda).$$

Why you should care

MOIs are a powerful tool in analysis. The identities we found for holomorphic functional calculus hold up for much larger classes of operators and functions:

Why you should care

MOIs are a powerful tool in analysis. The identities we found for holomorphic functional calculus hold up for much larger classes of operators and functions:

- $f(H+V) - f(H) = T^{H+V,H}_{f^{[1]}}(V)$ (analogous to Duhamel's formula);

Why you should care

MOIs are a powerful tool in analysis. The identities we found for holomorphic functional calculus hold up for much larger classes of operators and functions:

- $f(H+V) - f(H) = T^{H+V,H}_{f^{[1]}}(V)$ (analogous to Duhamel's formula);
- $[f(H), V] = T^{H,H}_{f^{[1]}}([H, V])$;

Why you should care

MOIs are a powerful tool in analysis. The identities we found for holomorphic functional calculus hold up for much larger classes of operators and functions:

- $f(H+V) - f(H) = T^{H+V,H}_{f^{[1]}}(V)$ (analogous to Duhamel's formula);
- $[f(H), V] = T^{H,H}_{f^{[1]}}([H,V])$;
- $\frac{d^n}{dt^n} f(H+tV)|_{t=0} = T^{H,\ldots,H}_{f^{[n]}}(V,\ldots,V)$,

each of which has been used to obtain sharp estimates. (Potapov, Sukochev, Skripka, Caspers, Montgomery-Smith, McDonald, Peller, ...)

Why you should care

MOIs are a powerful tool in analysis. The identities we found for holomorphic functional calculus hold up for much larger classes of operators and functions:

- $f(H+V) - f(H) = T^{H+V,H}_{f^{[1]}}(V)$ (analogous to Duhamel's formula);
- $[f(H), V] = T^{H,H}_{f^{[1]}}([H, V])$;
- $\frac{d^n}{dt^n} f(H+tV)|_{t=0} = T^{H,\ldots,H}_{f^{[n]}}(V, \ldots, V)$,

each of which has been used to obtain sharp estimates. (Potapov, Sukochev, Skripka, Caspers, Montgomery-Smith, McDonald, Peller, ...)

Furthermore, MOIs can systematise operator integral techniques in NCG.

Exhibit A

We use the Chern character of $(\mathcal{A}, \mathcal{H}, D)$ in entire cyclic cohomology (cf. [2]) given in the most efficient manner by the JLO formula, which defines the components of an entire cocycle in the (b, B) bicomplex:

(90) $$\psi_n(a^0, \ldots, a^n) = \sqrt{2i} \int_{\sum_0^n v_i = 1, v_i \geq 0}$$

$$\mathrm{Trace}\left(a^0\, e^{-v_0 D^2}\, [D, a^1]\, e^{-v_1 D^2} \ldots e^{-v_{n-1} D^2}\, [D, a^n]\, e^{-v_n D^2}\right), \qquad \forall a^j \in \mathcal{A}$$

where n is odd.

We introduce a parameter ϵ by replacing D^2 by ϵD^2, which yields a cocycle ψ_n^ϵ which is cohomologous to ψ_n. One has moreover

(91) $$\psi_n^\epsilon(a^0, \ldots, a^n) = \sqrt{2i} \left(\int_{\sum_0^n v_i = 1} \theta(\epsilon\, v_0, \ldots, \epsilon\, v_n)\, \pi\, dv_i \right) \epsilon^{n/2},$$

From [ConnesMoscovici1995]

Exhibit B

Let us now show that if $b \in \cap$ Dom $L^k R^q$ then $b \in$ Dom δ. The proof is more subtle than one would expect, because the obvious argument, using

$$|D| = \pi^{-1} \int_0^\infty \frac{D^2}{D^2+\mu}\, \mu^{-1/2}\, d\mu \ ,$$

requires some care. Indeed, one gets from the above

$$[|D|, b] = \pi^{-1} \int_0^\infty (D^2+\mu)^{-1}\, [D^2, b]\, (D^2+\mu)^{-1}\, \mu^{1/2}\, d\mu \ .$$

We can replace $[D^2, b]$ by $|D|$, which has the same size, and get

$$\int_0^\infty (D^2+\mu)^{-2}\, |D|\, \mu^{1/2}\, d\mu = \int_0^\infty (1+t)^{-2}\, t^{1/2}\, dt \ .$$

For this to work, we need to move $[D^2, b]$ in front of the above integral, i.e. use the finiteness of the norm of

$$\int_0^\infty \underbrace{[(D^2+\mu)^{-1}, [D^2, b]]}_{-(D^2+\mu)^{-1}[D^2,[D^2,b]](D^2+\mu)^{-1}}\ (D^2+\mu)^{-1}\, \mu^{1/2}\, d\mu \ .$$

This finiteness follows from:

1) $(D^2+\mu)^{-1}\, [D^2, [D^2, b]]$ bounded since $b \in \operatorname{Dom} L^2$

2) $\int_0^\infty \|(D^2+\mu)^{-2}\|\, \mu^{1/2}\, d\mu \le C \int_0^1 \mu^{1/2}\, d\mu + \int_1^\infty \mu^{-3/2}\, d\mu < \infty$.

Once $[D^2, b]$ is moved in front the above calculation applies.

From [ConnesMoscovici1995]

Exhibit C

Now, onwards with the computation, the first part of which is straightforward:

$$\begin{aligned}
[\Delta^{-z}, A]B &= \frac{1}{2\pi i}\int \lambda^{-z}[(\lambda-\Delta)^{-1}, A]B\, d\lambda \\
&= \frac{1}{2\pi i}\int \lambda^{-z}(\lambda-\Delta)^{-1}[\Delta, A](\lambda-\Delta)^{-1}B\, d\lambda \\
&= \int \lambda^{-z}(\lambda-\Delta)^{-1}[\Delta, A]B(\lambda-\Delta)^{-1}\, d\lambda \\
&\quad + \int \lambda^{-z}(\lambda-\Delta)^{-1}[\Delta, A](\lambda-\Delta)^{-1}[\Delta, B](\lambda-\Delta)^{-1}\, d\lambda.
\end{aligned}$$

(In the last step we did two things at once: we commuted B past $(\lambda-\Delta)^{-1}$ and we then used the formula $[S^{-1}, T] = S^{-1}[T, S]S^{-1}$.) The operators $[\Delta, A]$ and $[\Delta, B]$ have orders 1 and 2, respectively.

Before going on, we shall introduce some better notation for our contour integrals.

2.5 Definition. If $D_0, \ldots, D_p$ are differential operators on the closed manifold M, then denote by $I_z(D_0, \ldots, D_p)$ the integral

$$\frac{1}{2\pi i}\int \lambda^{-z}D_0(\lambda-\Delta)^{-1}\cdots D_p(\lambda-\Delta)^{-1}\, d\lambda$$

(in the integral, copies of $(\lambda-\Delta)^{-1}$ alternate with the operators D_j). The integral converges if $\mathrm{Re}(z) < n$, in the sense we discussed above, and defines an operator on $C^\infty(M)$.

From [Higson2003]

A problem

If you write, like in *The Local Index Formula in Noncommutative Geometry* by Nigel Higson, for a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and $a \in \mathcal{A}$,

$$\begin{aligned}
[D^{-2z}, a] &= \left[\int_\gamma \lambda^{-z}(\lambda - D^2)^{-1} d\lambda, a\right] \\
&= \int_\gamma \lambda^{-z}[(\lambda - D^2)^{-1}, a] d\lambda \\
&= \int_\gamma \lambda^{-z}(\lambda - D^2)^{-1}[D^2, a](\lambda - D^2)^{-1} d\lambda,
\end{aligned}$$

then $[D^2, a] \notin B(\mathcal{H})$, so this is not a standard MOI.

Part 2: Abstract pseudodifferential calculus

Pseudodifferential operators

On $\mathbb{R}^d$, a differential operator $L = \sum_{|\alpha| \le k} a_\alpha(x)\partial^\alpha$ can be written as

$$L = \mathcal{F}^{-1} \circ M_{p_L} \circ \mathcal{F},$$

where M_{p_L} indicates multiplying with the polynomial $p_L(x, \xi) := \sum_{|\alpha| \le k} a_\alpha(x)\xi^\alpha$.

Pseudodifferential operators

On $\mathbb{R}^d$, a differential operator $L = \sum_{|\alpha| \leq k} a_\alpha(x)\partial^\alpha$ can be written as

$$L = \mathcal{F}^{-1} \circ M_{p_L} \circ \mathcal{F},$$

where M_{p_L} indicates multiplying with the polynomial $p_L(x, \xi) := \sum_{|\alpha| \leq k} a_\alpha(x)\xi^\alpha$.

Generally speaking, a pseudodifferential operator of order k on $\mathbb{R}^d$ is an operator of the form $L = \mathcal{F}^{-1} \circ M_{p_L} \circ \mathcal{F}$ where the function p_L is more general, such that

$$L : \mathcal{H}^{s+k} \to \mathcal{H}^s,$$

where

$$\mathcal{H}^s(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}\big[(1 + |\xi|^2)^{s/2}\mathcal{F}f\big] \in L_2(\mathbb{R}^n)\},$$

are Bessel potential Sobolev spaces.

Pseudodifferential operators

On $\mathbb{R}^d$, a differential operator $L = \sum_{|\alpha|\leq k} a_\alpha(x)\partial^\alpha$ can be written as

$$L = \mathcal{F}^{-1} \circ M_{p_L} \circ \mathcal{F},$$

where M_{p_L} indicates multiplying with the polynomial $p_L(x,\xi) := \sum_{|\alpha|\leq k} a_\alpha(x)\xi^\alpha$.

Generally speaking, a pseudodifferential operator of order k on $\mathbb{R}^d$ is an operator of the form $L = \mathcal{F}^{-1} \circ M_{p_L} \circ \mathcal{F}$ where the function p_L is more general, such that

$$L : \mathcal{H}^{s+k} \to \mathcal{H}^s,$$

where

$$\mathcal{H}^s(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}\left[(1+|\xi|^2)^{s/2}\mathcal{F}f\right] \in L_2(\mathbb{R}^n)\},$$

are Bessel potential Sobolev spaces.

On a smooth manifold M, we can define classes of pseudodifferential operators on $L_2(M)$ by patching together operators like this.

Sobolev spaces

Given an invertible, positive self-adjoint operator Θ on a separable Hilbert space $\mathcal{H}$, we can define the 'Sobolev' spaces $\mathcal{H}^s$, $s \in \mathbb{R}$, as the completion of $\mathrm{dom}\Theta^s$ under the norm

$$\|\xi\|_s^2 = \langle \xi, \xi \rangle_s := \langle \Theta^s \xi, \Theta^s \xi \rangle_{\mathcal{H}} = \|\Theta^s \xi\|^2, \quad \xi \in \mathrm{dom}\Theta^s.$$

This forms a Hilbert space. We have continuous embeddings

$$\mathcal{H}^t \subseteq \mathcal{H}^s, \quad s \leq t,$$

because

$$\|\Theta^s \xi\| \leq \|\Theta^{s-t}\|_\infty \|\Theta^t \xi\|.$$

Sobolev spaces

Given an invertible, positive self-adjoint operator Θ on a separable Hilbert space $\mathcal{H}$, we can define the 'Sobolev' spaces $\mathcal{H}^s$, $s \in \mathbb{R}$, as the completion of $\mathrm{dom}\Theta^s$ under the norm

$$\|\xi\|_s^2 = \langle \xi, \xi \rangle_s := \langle \Theta^s \xi, \Theta^s \xi \rangle_{\mathcal{H}} = \|\Theta^s \xi\|^2, \quad \xi \in \mathrm{dom}\Theta^s.$$

This forms a Hilbert space. We have continuous embeddings

$$\mathcal{H}^t \subseteq \mathcal{H}^s, \quad s \leq t,$$

because

$$\|\Theta^s \xi\| \leq \|\Theta^{s-t}\|_\infty \|\Theta^t \xi\|.$$

We put

$$\mathcal{H}^\infty := \bigcap_{s \in \mathbb{R}} \mathcal{H}^s, \quad \mathcal{H}^{-\infty} := \bigcup_{s \in \mathbb{R}} \mathcal{H}^s,$$

and we get for free that $\mathcal{H}^\infty$ is dense in $\mathcal{H}$.

Analytic order

Even though Θ itself is an unbounded operator on $\mathcal{H}$, if we regard it as an operator

$$\Theta : \mathcal{H}^1 \to \mathcal{H}^0 = \mathcal{H},$$

it is a perfectly good bounded operator:

$$\|\Theta\|_{\mathcal{H}^1 \to \mathcal{H}^0} = \sup_{\xi : \|\Theta\xi\| \leq 1} \|\Theta\xi\| = 1.$$

Analytic order

Even though Θ itself is an unbounded operator on $\mathcal{H}$, if we regard it as an operator

$$\Theta : \mathcal{H}^1 \to \mathcal{H}^0 = \mathcal{H},$$

it is a perfectly good bounded operator:

$$\|\Theta\|_{\mathcal{H}^1 \to \mathcal{H}^0} = \sup_{\xi : \|\Theta\xi\| \le 1} \|\Theta\xi\| = 1.$$

We define $\mathrm{op}^r(\Theta)$ for $r \in \mathbb{R}$ as those $T : \mathcal{H}^\infty \to \mathcal{H}^\infty$ that extend to a bounded operator

$$T : \mathcal{H}^{s+r} \to \mathcal{H}^s, \quad s \in \mathbb{R}.$$

Even though Θ itself is an unbounded operator on $\mathcal{H}$, if we regard it as an operator

$$\Theta : \mathcal{H}^1 \to \mathcal{H}^0 = \mathcal{H},$$

it is a perfectly good bounded operator:

$$\|\Theta\|_{\mathcal{H}^1 \to \mathcal{H}^0} = \sup_{\xi : \|\Theta\xi\| \leq 1} \|\Theta\xi\| = 1.$$

We define $\mathrm{op}^r(\Theta)$ for $r \in \mathbb{R}$ as those $T : \mathcal{H}^\infty \to \mathcal{H}^\infty$ that extend to a bounded operator

$$T : \mathcal{H}^{s+r} \to \mathcal{H}^s, \quad s \in \mathbb{R}.$$

We define $\mathrm{OP}^r(\Theta)$ as those $T \in \mathrm{op}^r(\Theta)$ for which $\delta_\Theta^n(T) \in \mathrm{op}^r(\Theta)$ for all $n \in \mathbb{N}$, where $\delta_\Theta(T) = [\Theta, T]$.

Examples

This framework is the operator-theoretic skeleton of any typical (one-parameter) pseudodifferential calculus.

Examples

This framework is the operator-theoretic skeleton of any typical (one-parameter) pseudodifferential calculus.

- If Δ is the Laplace operator on $\mathbb{R}^n$, setting $\Theta = (1 - \Delta)^{1/2}$ gives the standard (Bessel potential) Sobolev spaces. The k-th order (pseudo)differential operators are contained in $\mathrm{OP}^k(\Theta)$.

Examples

This framework is the operator-theoretic skeleton of any typical (one-parameter) pseudodifferential calculus.

- If Δ is the Laplace operator on $\mathbb{R}^n$, setting $\Theta = (1 - \Delta)^{1/2}$ gives the standard (Bessel potential) Sobolev spaces. The k-th order (pseudo)differential operators are contained in $\mathrm{OP}^k(\Theta)$.
- Taking $\Theta = (1 - \Delta)^{1/2}$ where Δ is the sub-Laplacian on a stratified Lie group gives the Sobolev spaces defined by Folland and Stein.

Examples

This framework is the operator-theoretic skeleton of any typical (one-parameter) pseudodifferential calculus.

- If Δ is the Laplace operator on $\mathbb{R}^n$, setting $\Theta = (1-\Delta)^{1/2}$ gives the standard (Bessel potential) Sobolev spaces. The k-th order (pseudo)differential operators are contained in $\mathrm{OP}^k(\Theta)$.
- Taking $\Theta = (1-\Delta)^{1/2}$ where Δ is the sub-Laplacian on a stratified Lie group gives the Sobolev spaces defined by Folland and Stein.
- For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ it makes sense to put $\Theta = (1+D^2)^{1/2}$. Then for example $D \in \mathrm{OP}^1(\Theta)$, and for a *regular* spectral triple $a, [D, a] \in \mathrm{OP}^0(\Theta)$ for all $a \in \mathcal{A}$.

Examples

This framework is the operator-theoretic skeleton of any typical (one-parameter) pseudodifferential calculus.

- If Δ is the Laplace operator on $\mathbb{R}^n$, setting $\Theta = (1-\Delta)^{1/2}$ gives the standard (Bessel potential) Sobolev spaces. The k-th order (pseudo)differential operators are contained in $\mathrm{OP}^k(\Theta)$.
- Taking $\Theta = (1-\Delta)^{1/2}$ where Δ is the sub-Laplacian on a stratified Lie group gives the Sobolev spaces defined by Folland and Stein.
- For a spectral triple $(\mathcal{A}, \mathcal{H}, D)$ it makes sense to put $\Theta = (1+D^2)^{1/2}$. Then for example $D \in \mathrm{OP}^1(\Theta)$, and for a *regular* spectral triple $a, [D,a] \in \mathrm{OP}^0(\Theta)$ for all $a \in \mathcal{A}$.
- If Θ is bounded, $\mathcal{H}^s \simeq \mathcal{H}$ and $\mathrm{op}^r(\Theta) = B(\mathcal{H})$ for all $s, r \in \mathbb{R}$.

Elliptic operators

Our goal is to construct MOIs where all operators are in $\mathrm{op}(\Theta)$. First, we need a functional calculus for such operators. Analogously to usual notions of pseudodifferential operators, a functional calculus can be constructed for *elliptic* operators.

Elliptic operators

Our goal is to construct MOIs where all operators are in $\mathrm{op}(\Theta)$. First, we need a functional calculus for such operators. Analogously to usual notions of pseudodifferential operators, a functional calculus can be constructed for *elliptic* operators.

We define $T \in \mathrm{op}^r(\Theta)$ to be Θ-*elliptic*, if there is a parametrix $P \in \mathrm{op}^{-r}(\Theta)$ such that

$$TP = 1_{\mathcal{H}^\infty} + \mathrm{op}^{-\infty}(\Theta);$$
$$PT = 1_{\mathcal{H}^\infty} + \mathrm{op}^{-\infty}(\Theta).$$

By a Borel Lemma argument, it suffices if

$$TP = 1_{\mathcal{H}^\infty} + \mathrm{op}^{-1}(\Theta);$$
$$PT = 1_{\mathcal{H}^\infty} + \mathrm{op}^{-1}(\Theta).$$

Elliptic operators 2

For any spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and $\Theta = (1 + D^2)^{1/2}$, we have that $D \in \mathrm{op}^1(\Theta)$ is Θ-elliptic. Furthermore, $D + V$ is Θ-elliptic if $V \in \mathrm{op}^r(\Theta)$ with $r < 1$.

Elliptic operators 2

For any spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and $\Theta = (1 + D^2)^{1/2}$, we have that $D \in \mathrm{op}^1(\Theta)$ is Θ-elliptic. Furthermore, $D + V$ is Θ-elliptic if $V \in \mathrm{op}^r(\Theta)$ with $r < 1$.

If $T \in \mathrm{op}^r(\Theta)$ is Θ-elliptic,

- If $x \in \mathcal{H}^{-\infty}$, then $Tx \in \mathcal{H}^s$ implies that $x \in \mathcal{H}^{s+r}$ (*elliptic regularity*).
- If $T : \mathcal{H}^r \subseteq \mathcal{H}^0 \to \mathcal{H}^0$ (i.e. $r \geq 0$) is a symmetric operator, then it is self-adjoint given domain $\mathcal{H}^r$. This situation will be referred to as 'T is Θ-elliptic and symmetric'.

Elliptic operators 2

For any spectral triple $(\mathcal{A}, \mathcal{H}, D)$ and $\Theta = (1 + D^2)^{1/2}$, we have that $D \in \mathrm{op}^1(\Theta)$ is Θ-elliptic. Furthermore, $D + V$ is Θ-elliptic if $V \in \mathrm{op}^r(\Theta)$ with $r < 1$.

If $T \in \mathrm{op}^r(\Theta)$ is Θ-elliptic,

- If $x \in \mathcal{H}^{-\infty}$, then $Tx \in \mathcal{H}^s$ implies that $x \in \mathcal{H}^{s+r}$ (*elliptic regularity*).
- If $T : \mathcal{H}^r \subseteq \mathcal{H}^0 \to \mathcal{H}^0$ (i.e. $r \geq 0$) is a symmetric operator, then it is self-adjoint given domain $\mathcal{H}^r$. This situation will be referred to as 'T is Θ-elliptic and symmetric'. Note: this does *not* imply that

$$T : \mathcal{H}^{r+s} \subseteq \mathcal{H}^s \to \mathcal{H}^s$$

  is self-adjoint for any other $s \in \mathbb{R}$. In fact, these operators need not even be symmetric or normal.

Functional calculus

We write $f \in L^{\beta}_{\infty}(\mathbb{R}, E)$ for some $\beta \in \mathbb{R}$ and spectral measure E, if $f(x)(1+x^2)^{-\beta/2} \in L_{\infty}(\mathbb{R}, E)$.

Functional calculus

We write $f \in L_\infty^\beta(\mathbb{R}, E)$ for some $\beta \in \mathbb{R}$ and spectral measure E, if $f(x)(1+x^2)^{-\beta/2} \in L_\infty(\mathbb{R}, E)$.

H.–McDonald–van Nuland (2024)

Let $T \in \mathrm{op}^r(\Theta)$, $r > 0$, be Θ-elliptic and symmetric with spectral measure E_T. If $f \in L_\infty^\beta(\mathbb{R}, E_T)$, $\beta \in \mathbb{R}$, then

$$f(T) \in \mathrm{op}^{r\beta}(\Theta).$$

Functional calculus

We write $f \in L_\infty^\beta(\mathbb{R}, E)$ for some $\beta \in \mathbb{R}$ and spectral measure E, if $f(x)(1+x^2)^{-\beta/2} \in L_\infty(\mathbb{R}, E)$.

H.–McDonald–van Nuland (2024)

Let $T \in \mathrm{op}^r(\Theta)$, $r > 0$, be Θ-elliptic and symmetric with spectral measure E_T. If $f \in L_\infty^\beta(\mathbb{R}, E_T)$, $\beta \in \mathbb{R}$, then

$$f(T) \in \mathrm{op}^{r\beta}(\Theta).$$

Furthermore, if A is self-adjoint on $\mathcal{H}$, $A \in \mathrm{op}^t(\Theta)$, $t \in \mathbb{R}$, and A commutes strongly with T, then for $f \in L_\infty^\beta(\mathbb{R}, E_A)$, $\beta \geq 0$, we have

$$f(A) \in \mathrm{op}^{t\beta}(\Theta).$$

This second part applies for example to $i\frac{d}{dx}$ in $\mathrm{op}(1-\Delta)^{1/2}$ on $\mathbb{R}^d$.

Part 3: MOIs as pseudodifferential operators

H.–McDonald–van Nuland (2024)

Let $H_i \in \mathrm{op}^{h_i}(\Theta)$, $h_i > 0$ Θ-elliptic and symmetric for $i = 0, \ldots, n$, and $X_i \in \mathrm{op}^{r_i}(\Theta)$ for $i = 1, \ldots, n$. Let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ such that

$$\phi(\lambda_0, \ldots, \lambda_n) = \int_\Omega a_0(\lambda_0, \omega) \cdots a_n(\lambda_n, \omega) d\nu(\omega),$$

with finite measure space (Ω, ν) with $a_j(x, \omega)(1 + x^2)^{-\beta_j/2} : \mathbb{R} \times \Omega \to \mathbb{C}$ measurable and bounded. Then for $\psi \in \mathcal{H}^\infty$,

$$T_\phi^{H_0,\ldots,H_n}(X_1, \ldots, X_n)\psi := \int_\Omega a_0(H_0, \omega) X_1 a_1(H_1, \omega) \cdots X_n a_n(H_n, \omega) \psi d\nu(\omega)$$

is a well-defined vector in $\mathcal{H}^\infty$ independent of the representation of ϕ, and

$$T_\phi^{H_0,\ldots,H_n} : \mathrm{op}^{r_1}(\Theta) \times \cdots \times \mathrm{op}^{r_n}(\Theta) \to \mathrm{op}^{\sum_j r_j + \sum_j \beta_j h_j}(\Theta).$$

Unbounded MOIs: the useful bit

If $f \in C^{n+2}(\mathbb{R})$, and $f^{(k)} \in L_\infty^{\beta-k}(\mathbb{R})$ for $k = 0, \ldots, n+2$, then for $H \in \mathrm{op}^h(\Theta)$, $h > 0$ Θ-elliptic and symmetric, and $X_i \in \mathrm{op}^{r_i}(\Theta)$,

$$T^{H,\ldots,H}_{f^{[n]}}(X_1, \ldots, X_n) \in \bigcap_{\varepsilon>0} \mathrm{op}^{(\beta-n)h+\sum_j r_j+\varepsilon}(\Theta).$$

Note: the order is ε worse than that of $\frac{1}{n!}X_1 \cdots X_n f^{(n)}(H)$, which would equal the LHS if $X_1, \ldots, X_n$ commute strongly with H.

Unbounded MOIs: the useful bit

If $f \in C^{n+2}(\mathbb{R})$, and $f^{(k)} \in L_\infty^{\beta-k}(\mathbb{R})$ for $k = 0, \ldots, n+2$, then for $H \in \mathrm{op}^h(\Theta)$, $h > 0$ Θ-elliptic and symmetric, and $X_i \in \mathrm{op}^{r_i}(\Theta)$,

$$T^{H,\ldots,H}_{f^{[n]}}(X_1, \ldots, X_n) \in \bigcap_{\varepsilon>0} \mathrm{op}^{(\beta-n)h+\sum_j r_j+\varepsilon}(\Theta).$$

Note: the order is ε worse than that of $\frac{1}{n!}X_1 \cdots X_n f^{(n)}(H)$, which would equal the LHS if $X_1, \ldots, X_n$ commute strongly with H.

If $f \in C^\infty(\mathbb{R})$ and $f^{(k)} \in L_\infty^{\beta-k}(\mathbb{R})$ for all $k \in \mathbb{N}$, we write $f \in S^\beta(\mathbb{R})$.

Unbounded MOIs: the useful bit

If $f \in C^{n+2}(\mathbb{R})$, and $f^{(k)} \in L_\infty^{\beta-k}(\mathbb{R})$ for $k = 0, \ldots, n+2$, then for $H \in \mathrm{op}^h(\Theta)$, $h > 0$ Θ-elliptic and symmetric, and $X_i \in \mathrm{op}^{r_i}(\Theta)$,

$$T^{H,\ldots,H}_{f^{[n]}}(X_1, \ldots, X_n) \in \bigcap_{\varepsilon>0} \mathrm{op}^{(\beta-n)h+\sum_j r_j+\varepsilon}(\Theta).$$

Note: the order is ε worse than that of $\frac{1}{n!}X_1 \cdots X_n f^{(n)}(H)$, which would equal the LHS if $X_1, \ldots, X_n$ commute strongly with H.

If $f \in C^\infty(\mathbb{R})$ and $f^{(k)} \in L_\infty^{\beta-k}(\mathbb{R})$ for all $k \in \mathbb{N}$, we write $f \in S^\beta(\mathbb{R})$.

For integral we saw earlier, with the appropriate conditions on D and a, we have

$$\int_\gamma \lambda^{-z}(\lambda-D^2)^{-1}[D^2, a](D^2-\lambda)^{-1}d\lambda = T^{D^2,D^2}_{f^{[1]}}([D^2, a]) \in \bigcap_{\varepsilon>0} \mathrm{op}^{-2\Re(z)-1+\varepsilon}(\Theta),$$

with $f(x) = x^{-z}$ and $\Theta = (1 + D^2)^{\frac{1}{2}}$.

Two rules

MOIs as we defined them come with two identities:

1. $f(A) - f(B) = T_{f^{[1]}}^{A,B}(A - B)$;
2. $[f(H), a] = T_{f^{[1]}}^{H,H}([H, a])$,

Two rules

MOIs as we defined them come with two identities:

1. $f(A) - f(B) = T_{f^{[1]}}^{A,B}(A - B)$;
2. $[f(H), a] = T_{f^{[1]}}^{H,H}([H, a])$,

and the higher order analogues (since $T_{f^{[0]}}^{H}() = f(H)$)

1. $T_{f^{[n]}}^{H_0,\dots,A,\dots,H_n}(V_1, \dots, V_n) - T_{f^{[n]}}^{H_0,\dots,B,\dots,H_n}(V_1, \dots, V_n)$
   $= T_{f^{[n+1]}}^{H_0,\dots,A,B,\dots,H_n}(V_1, \dots, A - B, \dots, V_n)$;

2. $T_{f^{[n]}}^{H_0,\dots,H_n}(V_1, \dots, V_{j-1}, aV_j, \dots, V_n) - T_{f^{[n]}}^{H_0,\dots,H_n}(V_1, \dots, V_{j-1}a, V_j, \dots, V_n)$
   $= T_{f^{[n+1]}}^{H_0,\dots,H_j,H_j,\dots,H_n}(V_1, \dots, V_{j-1}, [H_j, a], V_{j+1}, \dots, V_n)$.

Taylor expansion

The first rule on its own gives a Taylor expansion:

$$\begin{aligned} f(H+V) &\overset{(1)}{=} f(H) + T_{f^{[1]}}^{H+V,H}(V) \\ &\overset{(1)}{=} f(H) + T_{f^{[1]}}^{H,H}(V) + T_{f^{[2]}}^{H+V,H,H}(V,V), \end{aligned}$$

and repeat. We get for all $N \in \mathbb{N}$

$$f(H+V) = \sum_{n=0}^{N} T_{f^{[n]}}^{H,\ldots,H}(V,\ldots,V) + T_{f^{[N+1]}}^{H+V,H,\ldots,H}(V,\ldots,V).$$

Note: if H and V commute,

$$T_{f^{[n]}}^{H,\ldots,H}(V,\ldots,V) = \frac{1}{n!} f^{(n)}(H) V^n.$$

Asymptotic expansions

We say that $T \sim \sum_{k=0}^{\infty} T_k$ for $T, T_k \in \mathrm{op}(\Theta)$ if

$$T - \sum_{k=1}^{N} T_k \in \mathrm{op}^{m_N}(\Theta), \quad m_N \downarrow -\infty.$$

Asymptotic expansions

We say that $T \sim \sum_{k=0}^{\infty} T_k$ for $T, T_k \in \mathrm{op}(\Theta)$ if

$$T - \sum_{k=1}^{N} T_k \in \mathrm{op}^{m_N}(\Theta), \quad m_N \downarrow -\infty.$$

If $f \in S^{\beta}(\mathbb{R})$, if $H \in \mathrm{op}^h(\Theta)$, $h > 0$ is Θ-elliptic and symmetric, and if $V \in \mathrm{op}^r(\Theta)$ with $r < h$, then

$$f(H+V) \sim \sum_{n=0}^{\infty} T_{f^{[n]}}^{H,\dots,H}(V,\dots,V).$$

If furthermore $\delta_H^n(V) \in \mathrm{op}^{r+n(h-\varepsilon)}$ for some $\varepsilon > 0$ (for example $H = \Theta$, $V \in \mathrm{OP}^r(\Theta)$)

$$T_{f^{[n]}}^{H,\dots,H}(V,\dots,V) \sim \sum_{m=0}^{\infty} \sum_{m_1+\dots+m_n=m} \frac{C_{m_1,\dots,m_n}}{(n+m)!} \delta_H^{m_1}(V)\cdots\delta_H^{m_n}(V) f^{(n+m)}(H).$$

Full expansion

Combining these two asymptotic expansions (if H and V satisfy both sets of conditions), we obtain

$$f(H+V) \sim \sum_{n,m=0}^{\infty} \sum_{m_1+\cdots+m_n=m} \frac{C_{m_1,\ldots,m_n}}{(n+m)!} \delta_H^{m_1}(V) \cdots \delta_H^{m_n}(V) f^{(n+m)}(H).$$

Note: this was obtained recognised for PsDOs on closed manifolds in [Paycha2006].

Part 4: Noncommutative QFT

QFT in a nutshell

Quantum Field Theory is infamously *not mathematically rigorous*.

QFT in a nutshell

Quantum Field Theory is infamously *not mathematically rigorous*.

Its ingredients are:

- Objects called fields ϕ, which are in the best case functions $\phi : M \to \mathbb{C}$ on space-time M, or worse, operator-valued distributions on M;
- An *action* $S[\phi]$ which is a functional on the space of fields. This action is expressed in terms of a Lagrangian $\mathcal{L}$ through $S[\phi] = \int_M \mathcal{L}[\phi]\, d\nu$.

For an observable F, also a functional on the space of fields, its expected value is given by the path integral

$$\langle F \rangle = \frac{\int_{\text{fields}} F[\phi] e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}\phi}{\int_{\text{fields}} e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}\phi}.$$

Standard model

$$\mathcal{L}_{SM} = -\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu + \frac{1}{2}ig_s^2(\bar{q}^\sigma_i \gamma^\mu q^\sigma_j)g^a_\mu + \bar{G}^a\partial^2 G^a + g_s f^{abc}\partial_\mu \bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \frac{1}{2c_w^2}M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \frac{1}{2}\partial_\mu H \partial_\mu H - \frac{1}{2}m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+\phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \frac{1}{2c_w^2}M\phi^0\phi^0 - \beta_h[\frac{2M^2}{g^2} + \frac{2M}{g}H + \frac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)] + \frac{2M^4}{g^2}\alpha_h - igc_w[\partial_\nu Z^0_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + Z^0_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)] - igs_w[\partial_\nu A_\mu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu(W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu(W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)] - \frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2 c_w^2(Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2(A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w[A_\mu Z^0_\nu(W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha[H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] - \frac{1}{8}g^2\alpha_h[H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2 H^2] - gMW^+_\mu W^-_\mu H - \frac{1}{2}g\frac{M}{c_w^2}Z^0_\mu Z^0_\mu H - \frac{1}{2}ig[W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)] + \frac{1}{2}g[W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)] + \frac{1}{2}g\frac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 - \phi^0\partial_\mu H) - ig\frac{s_w^2}{c_w}MZ^0_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) + igs_w MA_\mu(W^+_\mu\phi^- - W^-_\mu\phi^+) - ig\frac{1-2c_w^2}{2c_w}Z^0_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) + igs_w A_\mu(\phi^+\partial_\mu\phi^- - \phi^-\partial_\mu\phi^+) - \frac{1}{4}g^2 W^+_\mu W^-_\mu[H^2 + (\phi^0)^2 + 2\phi^+\phi^-] - \frac{1}{4}g^2\frac{1}{c_w^2}Z^0_\mu Z^0_\mu[H^2 + (\phi^0)^2 + 2(2s_w^2 -$$

Gaussian integrals

A fundamental technique to calculate the path integrals in QFT is inspired by the calculation that for $A \in M_n(\mathbb{C})$ invertible such that $A = A^T$, $\Re(A) \geq 0$, and $i_1, \ldots, i_k \in \{1, \ldots, n\}$,

$$\int_{\mathbb{R}^n} x_{i_1} \cdots x_{i_k} e^{-\frac{1}{2}\langle x, Ax\rangle} d^n x = \begin{cases} 0 & \text{k odd,} \\ \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \sum (A^{-1})_{i_{j_1} i_{j_2}} \cdots (A^{-1})_{i_{j_{k-1}} i_{j_k}} & \text{k even,} \end{cases}$$

where the sum is taken over all ways of grouping the indices $i_1, \ldots, i_k$ into $k/2$ unordered pairs.

Feynman graphs

The result of the previous computation can be expressed with graphs. Suppose for example that we are computing $\int_{\mathbb{R}^n} x_{i_1}^3 x_{i_2}^5 \, e^{-\frac{1}{2}\langle x, Ax\rangle} \, d^n x$, i.e. the indices $i_1, \ldots, i_k$ are given by $i_1, i_1, i_1, i_2, i_2, i_2, i_2, i_2$.

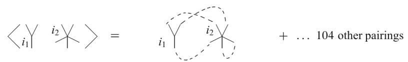
Feynman graphs

The result of the previous computation can be expressed with graphs. Suppose for example that we are computing $\int_{\mathbb{R}^n} x_{i_1}^3 x_{i_2}^5 \, e^{-\frac{1}{2}\langle x, Ax\rangle} \, d^n x$, i.e. the indices $i_1, \dots, i_k$ are given by $i_1, i_1, i_1, i_2, i_2, i_2, i_2, i_2$. As $k = 8$ is even, we need to consider all ways of grouping these indices into 4 unordered pairs.

Feynman graphs

The result of the previous computation can be expressed with graphs. Suppose for example that we are computing $\int_{\mathbb{R}^n} x_{i_1}^3 x_{i_2}^5 \, e^{-\frac{1}{2}\langle x, Ax\rangle} \, d^n x$, i.e. the indices $i_1, \ldots, i_k$ are given by $i_1, i_1, i_1, i_2, i_2, i_2, i_2, i_2$. As $k = 8$ is even, we need to consider all ways of grouping these indices into 4 unordered pairs.

Each choice of such a grouping corresponds with a graph,



Amplitudes

Hence, for even k,

$$\int_{\mathbb{R}^n} x_{i_1} \cdots x_{i_k} e^{-\frac{1}{2}\langle x, Ax\rangle} d^n x = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \sum_{\text{graphs } \Gamma} \mathrm{Ampl}(\Gamma),$$

where the sum is taken over all graphs Γ with vertices given by the indices present on the left-hand side, the degree of the vertices are the multiplicities of the corresponding indices on the left-hand side, and

$$\mathrm{Ampl}(\Gamma) = \prod_{e \text{ edges in } \Gamma} (A^{-1})_{i_e j_e}.$$

Hence, for even k,

$$\int_{\mathbb{R}^n} x_{i_1}\cdots x_{i_k} e^{-\frac{1}{2}\langle x, Ax\rangle} d^n x = \frac{(2\pi)^{n/2}}{\sqrt{\det A}} \sum_{\text{graphs } \Gamma} \mathrm{Ampl}(\Gamma),$$

where the sum is taken over all graphs Γ with vertices given by the indices present on the left-hand side, the degree of the vertices are the multiplicities of the corresponding indices on the left-hand side, and

$$\mathrm{Ampl}(\Gamma) = \prod_{e \text{ edges in } \Gamma} (A^{-1})_{i_e j_e}.$$

The path integral ‘formalism’ in QFT is analogous to this computation applied to $\frac{\int_{\mathrm{fields}} F[\phi] e^{\frac{1}{\hbar}S[\phi]}\mathcal{D}\phi}{\int_{\mathrm{fields}} e^{\frac{1}{\hbar}S[\phi]}\mathcal{D}\phi}$, but on steroids (and not mathematically rigorous).

Power counting

On the nose, the integrals $\frac{\int_{\mathrm{fields}} F[\phi] e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}\phi}{\int_{\mathrm{fields}} e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}\phi}$ will typically diverge.

Power counting

On the nose, the integrals $\frac{\int_{\text{fields}} F[\phi] e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}\phi}{\int_{\text{fields}} e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}\phi}$ will typically diverge.

A common first step to ‘renormalisation’ in QFT is to introduce a regularising parameter N (or ε), so that formally

$$\frac{\int_{\text{fields}(N)} F[\phi] e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}_N\phi}{\int_{\text{fields}(N)} e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}_N\phi} \xrightarrow{N\to\infty} \frac{\int_{\text{fields}} F[\phi] e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}\phi}{\int_{\text{fields}} e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}\phi},$$

and

$$\frac{\int_{\text{fields}(N)} F[\phi] e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}_N\phi}{\int_{\text{fields}(N)} e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}_N\phi} = \sum_{\text{graphs } \Gamma} \mathrm{Ampl}(\Gamma, N).$$

Power counting

On the nose, the integrals $\frac{\int_{\text{fields}} F[\phi] e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}\phi}{\int_{\text{fields}} e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}\phi}$ will typically diverge.

A common first step to 'renormalisation' in QFT is to introduce a regularising parameter N (or ε), so that formally

$$\frac{\int_{\text{fields}(N)} F[\phi] e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}_N\phi}{\int_{\text{fields}(N)} e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}_N\phi} \xrightarrow{N\to\infty} \frac{\int_{\text{fields}} F[\phi] e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}\phi}{\int_{\text{fields}} e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}\phi},$$

and

$$\frac{\int_{\text{fields}(N)} F[\phi] e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}_N\phi}{\int_{\text{fields}(N)} e^{\frac{1}{\hbar} S[\phi]} \mathcal{D}_N\phi} = \sum_{\text{graphs } \Gamma} \mathrm{Ampl}(\Gamma, N).$$

'Power counting' is the analysis of the divergence of $\mathrm{Ampl}(\Gamma, N)$ as $N \to \infty$.

By postulating that space-time is given by a very clever spectral triple $(\mathcal{A}, \mathcal{H}, D)$, Connes and Chamseddine managed to derive the bosonic part of the Lagrangian of the standard model from the asymptotic expansion as $\Lambda \to \infty$ of

$$\mathrm{Tr}\Big(f\Big(\frac{D+V}{\Lambda}\Big)\Big), \quad f \in C^\infty(\mathbb{R}).$$

Spectral action

By postulating that space-time is given by a very clever spectral triple $(\mathcal{A}, \mathcal{H}, D)$, Connes and Chamseddine managed to derive the bosonic part of the Lagrangian of the standard model from the asymptotic expansion as $\Lambda \to \infty$ of

$$\mathrm{Tr}\left(f\left(\frac{D+V}{\Lambda}\right)\right), \quad f \in C^{\infty}(\mathbb{R}).$$

Continuing work by van Nuland and van Suijlekom, one can instead try to approach Quantum Field Theory by considering the spectral action functional $V \mapsto \mathrm{Tr}(f(D+V))$ *itself* as the functional of the physical theory, instead of taking its asymptotic expansion first.

A basic noncommutative QFT model

For D a self-adjoint operator with compact resolvent on a separable Hilbert space $\mathcal{H}$, we study

$$\frac{\int_{\text{gauge fields}} V_{i_1 j_1} \cdots V_{i_m j_m} e^{\frac{1}{\hbar}\mathrm{Tr}(f(D+V)-f(D))}\, d[V]}{\int_{\text{gauge fields}} e^{\frac{1}{\hbar}\mathrm{Tr}(f(D+V)-f(D))}\, d[V]}.$$

The space of gauge fields is given by

$$\Omega^1_D(\mathcal{A})_{sa} = \Big\{ \sum_{j=1}^{N} a_j [D, b_j] \in B(\mathcal{H})_{sa} : N \geq 0, a_j, b_j \in \mathcal{A} \Big\}.$$

A basic noncommutative QFT model

For D a self-adjoint operator with compact resolvent on a separable Hilbert space $\mathcal{H}$, we study

$$\frac{\int_{\text{gauge fields}} V_{i_1 j_1} \cdots V_{i_m j_m} e^{\frac{1}{\hbar}\mathrm{Tr}(f(D+V)-f(D))}\, d[V]}{\int_{\text{gauge fields}} e^{\frac{1}{\hbar}\mathrm{Tr}(f(D+V)-f(D))}\, d[V]}.$$

The space of gauge fields is given by

$$\Omega^1_D(\mathcal{A})_{sa} = \Big\{ \sum_{j=1}^N a_j[D, b_j] \in B(\mathcal{H})_{sa} : N \geq 0, a_j, b_j \in \mathcal{A} \Big\}.$$

As a very basic regularisation, we can simply study

$$\frac{\int_{H_N} V_{i_1 j_1} \cdots V_{i_m j_m} e^{\frac{1}{\hbar}\mathrm{Tr}(f(D+V)-f(D))}\, d[V]}{\int_{H_N} e^{\frac{1}{\hbar}\mathrm{Tr}(f(D+V)-f(D))}\, d[V]},$$

where $H_N = \mathrm{span}\{|\psi_i\rangle\langle\psi_j| : 1 \leq i, j \leq N\}_{sa}$ in an eigenbasis of D and

$$\int_{H_N} d[V] := \Bigg(\prod_{1 \leq j \leq k \leq N} \int_{\mathbb{R}} d(\Re(V_{jk})) \Bigg) \Bigg(\prod_{1 \leq i < j \leq N} \int_{\mathbb{R}} d(\Im(V_{jk})) \Bigg).$$

Feynman diagrams

Assuming that $f \in C_c^\infty(\mathbb{R})$ and $V \in H_N$, we use that

$$\begin{aligned}
\mathrm{Tr}(f(D+V) - f(D)) &= \sum_{n=1}^{\infty} \mathrm{Tr}(T_{f^{[n]}}^{D,\ldots,D}(V,\ldots,V)) \\
&= \sum_{n=1}^{\infty} \frac{1}{n} \mathrm{Tr}(V T_{(f')^{[n-1]}}^{D,\ldots,D}(V,\ldots,V)) \\
&= \sum_{n=1}^{\infty} \sum_{i_1,\ldots,i_n=1}^{N} \frac{1}{n} (f')^{[n-1]} V_{i_1 i_2} \cdots V_{i_n i_1}.
\end{aligned}$$

Assuming that $f \in C_c^\infty(\mathbb{R})$ and $V \in H_N$, we use that

$$\begin{aligned}\mathrm{Tr}(f(D+V)-f(D)) &= \sum_{n=1}^{\infty} \mathrm{Tr}(T_{f^{[n]}}^{D,\dots,D}(V,\dots,V)) \\ &= \sum_{n=1}^{\infty} \frac{1}{n}\mathrm{Tr}(VT_{(f')^{[n-1]}}^{D,\dots,D}(V,\dots,V)) \\ &= \sum_{n=1}^{\infty} \sum_{i_1,\dots,i_n=1}^{N} \frac{1}{n}(f')^{[n-1]}V_{i_1 i_2}\cdots V_{i_n i_1}.\end{aligned}$$

It was found by van Nuland and van Suijlekom that this allows one, using (rigorous!) Gaussian integral techniques, to write

$$\frac{\int_{H_N} V_{i_1 j_1}\cdots V_{i_m j_m} e^{\frac{1}{\hbar}\mathrm{Tr}(f(D+V)-f(D))}\, d[V]}{\int_{H_N} e^{\frac{1}{\hbar}\mathrm{Tr}(f(D+V)-f(D))}\, d[V]} = \sum_{\text{ribbon graphs } \Gamma} \mathrm{Ampl}(\Gamma, N),$$

where the sum is taken over a certain kind of *ribbon graphs*.

Ribbon graphs

The graphs appearing in the expansion are given by (G_0, n, G_1) where G_0 is a set of *internal vertices*, $n \in \mathbb{N}$ is the number of *external vertices* (each with degree 1, decorated by $\{1, \ldots, n\}$), G_1 is the set of edges, and where each internal vertex has a cyclic ordering.

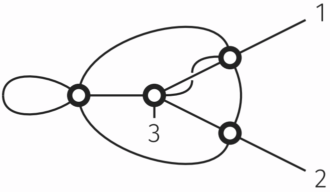


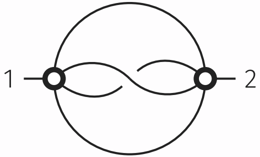
Figure 1: A Feynman ribbon graph $(G_0, 3, G_1)$.

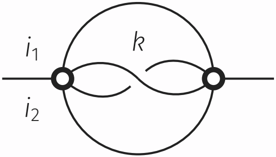
Ribbon graph amplitudes (I)

Given a Feynman ribbon graph $G = (G_0, n, G_1)$, the relevant amplitudes we need to calculate are given by a choice $i_1, \ldots, i_n \in \mathbb{N}$,

$$\mathrm{Ampl}^{f,D}_{N,i_1,\ldots,i_n}(G),$$

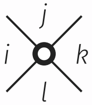
using some combinatorial procedure. As an example,

$\mathrm{Ampl}^{f,D}_{N,i_1,i_2}\Bigg($ 

$\Bigg) = \sum_{k=1}^{N}$ 

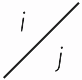
$$= \sum_{k=1}^{N} \frac{f'[\lambda_{i_1}, \lambda_k, \lambda_k, \lambda_k, \lambda_{i_2}]^2}{f'[\lambda_{i_1}, \lambda_k] f'[\lambda_k, \lambda_k]^2 f'[\lambda_k, \lambda_{i_2}]}.$$

For each vertex bordered by faces with indices $i_1, \ldots, i_n$, we multiply by a factor $f'[\lambda_{i_1}, \ldots, \lambda_{i_n}]$, e.g.,



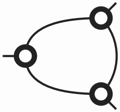
$$= \quad f'[\lambda_i, \lambda_j, \lambda_k, \lambda_l],$$

for each internal edge bordered by i and j, we divide by a factor $f'[\lambda_i, \lambda_j]$, i.e.,

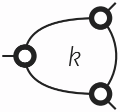


$$= \quad \frac{1}{f'[\lambda_i, \lambda_j]},$$

and, finally, we sum over each unbroken face (face without external edges), e.g.,



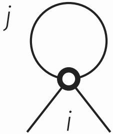
$$= \quad \sum_{k=1}^{N}$$



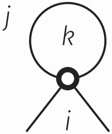
.

Examples

Two amplitudes in the spectral action matrix model are

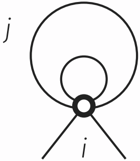


$$= \sum_{k=1}^{N}$$

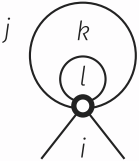


$$= \sum_{k=1}^{N} \frac{f'[\lambda_j, \lambda_k, \lambda_j, \lambda_i]}{f'[\lambda_j, \lambda_k]},$$

and



$$= \sum_{k,l=1}^{N}$$



$$= \sum_{k,l=1}^{N} \frac{f'[\lambda_j, \lambda_k, \lambda_l, \lambda_k, \lambda_j, \lambda_i]}{f'[\lambda_j, \lambda_k] f'[\lambda_k, \lambda_l]}.$$

Divided differences

Together with Teun van Nuland and Jesse Reimann, we proved some delicate estimates on divided differences. We say that $f \in C^\infty(\mathbb{R})$ is **of precise order** $r \in \mathbb{R}$ if for all $k \in \mathbb{N}$ there exist positive numbers $R, c_1, c_2 > 0$ (depending on k), such that

$$c_1|x|^{r-k} \leq |f^{(k)}(x)| \leq c_2|x|^{r-k} \tag{1}$$

for all $x \in \mathbb{R} \setminus [-R, R]$.

Theorem (H.–van Nuland–Reimann)

Let $f \in C^\infty(\mathbb{R})$ be such that f' is of precise order $-p-1$ for some $p \geq 0$, and f is positive outside a compact region. Let $n \in \mathbb{N}_{\geq 1}$ and assume that $\{\lambda_k\}_{k=1}^\infty \subseteq \mathbb{R} \setminus \{0\}$ is such that $f'[\lambda_{k_1}, \ldots, \lambda_{k_m}] \neq 0$ for every $m = 1, \ldots, n$ and $k_1, \ldots, k_m \in \mathbb{N}_{\geq 1}$. There exist $c_1, c_2 > 0$ such that, for all $k_1, \ldots, k_n$ with $|\lambda_{k_1}| \leq \cdots \leq |\lambda_{k_n}|$, we have

$$c_1|\lambda_{k_1}|^{-p} \leq (-1)^n \lambda_{k_1} \cdots \lambda_{k_n} f'[\lambda_{k_1}, \ldots, \lambda_{k_n}] \leq c_2|\lambda_{k_1}|^{-p}.$$

Theorem (H.–van Nuland–Reimann)

Let $f \in C^\infty(\mathbb{R})_\mathbb{R}$ be an even function such that f' is of precise order $-p-1$ for some $p \in \mathbb{R}_{\geq 0}$. Let D have an eigenvalue sequence for which there exist constants $K, c_1, c_2 > 0$ such that for all $k \geq K$ we have $c_1 k^{1/d} \leq |\lambda_k| \leq c_2 k^{1/d}$. Assume $f'[\lambda_{k_1}, \ldots, \lambda_{k_n}] \neq 0$ for all $k_1, \ldots, k_n \in \mathbb{N}_{\geq 1}$ and all $n \in \mathbb{N}_{\geq 1}$. Then for any Feynman ribbon graph $G = (G^0, n, G^1)$ whose vertices have valence ≥ 3, for all external indices $i_1, \ldots, i_n \in \mathbb{N}_{\geq 1}$, there exist $M, c_3, c_4 > 0$ such that, for all $N \geq M$,

$$c_3 N^{\omega(G)} \leq |\mathrm{Ampl}^{f,D}_{N,i_1,\ldots,i_n}(G)| \leq c_4 N^{\omega(G)};$$

$$\omega(G) = U + \frac{p}{d}(E_{\text{fi}} - V_{\text{fi}}),$$

where U is the number of unbroken faces of G, E_{fi} is the number of fully internal edges of G (propagators bordered on both sides by unbroken faces) and V_{fi} is the number of fully internal vertices of G.

Nonzero-condition

The assumption that $f'[\lambda_{k_1}, \ldots, \lambda_{k_n}] \neq 0$ for all $k_1, \ldots, k_n \in \mathbb{N}_{\geq 1}$ and all $n \in \mathbb{N}_{\geq 1}$ is admittedly quite restrictive, but important.

Nonzero-condition

The assumption that $f'[\lambda_{k_1}, \ldots, \lambda_{k_n}] \neq 0$ for all $k_1, \ldots, k_n \in \mathbb{N}_{\geq 1}$ and all $n \in \mathbb{N}_{\geq 1}$ is admittedly quite restrictive, but important.

This comes into play in asymptotics for essentially the following reason. If f has precise order $r < 0$, then for x fixed with $f(x) \neq 0$ and $y \to \infty$, we have

$$f^{[1]}(x, y) = \frac{f(x) - f(y)}{x - y} \approx -\frac{f(x)}{y} = O(y^{-1}).$$

But, if $f(x) = 0$, the asymptotic is more like

$$f^{[1]}(x, y) = \frac{-f(y)}{x - y} \approx -\frac{f(y)}{y} = O(y^{r-1}).$$

Given a subset $\mathfrak{b} \subseteq \mathcal{U}$ of the set of unbroken faces of a Feynman ribbon graph G, we let $G_{\mathfrak{b}}$ be the graph obtained from G by artificially declaring the faces in $\mathfrak{b}$ to be broken. Elements of $\mathfrak{b}$ are called 0-faces.

Theorem (H.–van Nuland–Reimann)

Conditions as before, but without assuming $f'[\lambda_{k_1}, \ldots, \lambda_{k_n}] \neq 0$. Then, for all external indices $i_1, \ldots, i_n \in \mathbb{N}_{\geq 1}$, there exist $M, c_4 > 0$ such that for all $N \geq M$,

$$|\mathrm{Ampl}_{N,i_1,\ldots,i_n}(G)| \leq c_4 N^{\tilde{\omega}(G)},$$

$$\tilde{\omega}(G) := \max_{\mathfrak{b} \subseteq \mathcal{U}} \omega_{\mathfrak{b}}(G_{\mathfrak{b}}) := \max_{\mathfrak{b} \subseteq \mathcal{U}} (U^{\mathfrak{b}} + \frac{p}{d}(E^{\mathfrak{b}}_{\mathrm{fi}} - V^{\mathfrak{b}}_{\mathrm{fi}}) + \frac{p+1}{d}(E^{\mathfrak{b}}_{10} - V^{\mathfrak{b}}_{10})),$$

where $U^{\mathfrak{b}}$ is the number of unbroken faces of $G_{\mathfrak{b}}$, $E^{\mathfrak{b}}_{\mathrm{fi}}$ is the number of fully internal edges of $G_{\mathfrak{b}}$ and $V^{\mathfrak{b}}_{\mathrm{fi}}$ is the number of fully internal vertices of $G_{\mathfrak{b}}$. Respectively, $E^{\mathfrak{b}}_{10}$ and $V^{\mathfrak{b}}_{10}$ are the number of edges and vertices of $G_{\mathfrak{b}}$ that border exactly one 0-face and for the rest unbroken faces.

A future project...

In the future, for operators $V_1, \ldots, V_n$, we will need to study the behaviour of expressions

$$\sum_{i.j=1}^{N} (V_1)_{i_1 i_2} \cdots (V_n)_{i_n i_1} \mathrm{Ampl}^{f,D}_{N, i_1, \ldots, i_n}(G). \tag{2}$$

A future project...

In the future, for operators $V_1, \ldots, V_n$, we will need to study the behaviour of expressions

$$\sum_{i.j=1}^{N} (V_1)_{i_1 i_2} \cdots (V_n)_{i_n i_1} \mathrm{Ampl}^{f,D}_{N,i_1,\ldots,i_n}(G). \tag{2}$$

Between the combination of abstract pseudodifferential calculus, multiple operator integrals and our previous work on divided differences, we are optimistic these divergences can be determined too.

Thanks

Thanks for your attention!