The Joy of MOIs: unbounded multiple operator integrals in noncommutative geometry

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Summary of this talk

1. MOIs in a nutshell
2. Pseudodifferential calculus
3. MOIs as pseudodifferential operators

This talk is based on work in progress with Teun van Nuland.

PhD supervisors: Fedor Sukochev (main), Edward McDonald and Dmitriy Zanin.

Spectral action

Describing nature by use of a self-adjoint operator D, the spectrum of D contains much information about the physics of the system. A good way of studying the spectrum and these physics is by considering

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or

$$\mathrm{Tr}(f(D+V)),$$

for a suitable function $f : \mathbb{R} \to \mathbb{C}$, and a suitable self-adjoint operator V.

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for a suitable function $f : \mathbb{R} \to \mathbb{C}$, and a suitable self-adjoint operator V.

For example, describing the universe as a carefully constructed spectral triple $(\mathcal{A}, \mathcal{H}, D)$, the entire Lagrangian of the standard model can be derived in this manner.

Taylor expansion

If f is smooth enough, a naive way to analyse $\mathrm{Tr}(f(D+V))$ is to perform a Taylor expansion of the function $t \mapsto \mathrm{Tr}(f(D+tV))$ around $t=0$:

$$\begin{aligned}\mathrm{Tr}(f(D+V)) &=^{?} \sum_{n=0}^{\infty} \frac{1}{n!}\frac{d^n}{dt^n}\Big|_{t=0} \mathrm{Tr}(f(D+tV)) \\ &=^{!?} \sum_{n=0}^{\infty} \mathrm{Tr}\left(\frac{1}{n!}\frac{d^n}{dt^n}\Big|_{t=0} f(D+tV)\right).\end{aligned}$$

What on earth is $\frac{1}{n!}\frac{d^n}{dt^n}\big|_{t=0} f(D+tV)$?

Part 1: MOIs in a nutshell

Origins of the MOI

Suppose we have two self-adjoint operators A, B. Can we make sense of $\frac{d^n}{dt^n}\big|_{t=0} f(A+tB)$? For which functions f and which restrictions on A and B?

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Yes, MOIs can help us understand. For $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$, $V_1, \dots, V_n \in B(\mathcal{H})$ and self-adjoint operators $H_0, \dots, H_n$ with spectral measures dE_j we define corresponding multiple operator integral as

$$T_\phi^{H_0,\dots,H_n}(V_1, \dots, V_n) := \int_{\sigma(H_0)\times\cdots\times\sigma(H_n)} \phi(\lambda_0, \dots, \lambda_n) dE_0(\lambda_0) V_1 dE_1(\lambda_1) \cdots V_n dE_n(\lambda_n).$$

Then for possibly unbounded self-adjoint A, bounded self-adjoint B,

$$\frac{d^n}{dt^n}\bigg|_{t=0} f(A+tB) = T_{f^{[n]}}^{A,\dots,A}(\underbrace{B, \dots, B}_{n}),$$

if the right hand side is defined.

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Then for possibly unbounded self-adjoint A, bounded self-adjoint B,

$$\frac{d^n}{dt^n}\bigg|_{t=0} f(A+tB) = T_{f^{[n]}}^{A,\ldots,A}(\underbrace{B,\ldots,B}_{n}),$$

if the right hand side is defined. Note: for $n=0$ we have $T_f^A() = f(A)$.

The MOI game

To understand $\frac{d^n}{dt^n}\Big|_{t=0} f(A+tB)$, we now have to determine when

$$T_\phi^{H_0,\ldots,H_n}(V_1,\ldots,V_n) = \int_{\sigma(H_0)\times\cdots\times\sigma(H_n)} \phi(\lambda_0,\ldots,\lambda_n)\,dE_0(\lambda_0)V_1dE_1(\lambda_1)\cdots V_ndE_n(\lambda_n)$$

is well-defined.

From a connection with Schur multiplier theory, for $\phi \in L_\infty$ we have that

$$T_\phi^{H_0,\ldots,H_n} : \mathcal{L}_2 \times \cdots \times \mathcal{L}_2 \to \mathcal{L}_2$$

is well-defined.

Separation of variables

As we have already seen, if we can write

$$\phi(\lambda_0, \dots, \lambda_n) = \int_\Omega a_0(\lambda_0, \omega) \cdots a_n(\lambda_n, \omega) d\nu(\omega),$$

then

$$\begin{aligned} &T_\phi^{H_0,\dots,H_n}(V_1, \dots, V_n) \\ &= \int_{\sigma(H_0)\times\cdots\times\sigma(H_n)} \phi(\lambda_0, \dots, \lambda_n) dE_0(\lambda_0) V_1 dE_1(\lambda_1) \cdots V_n dE_n(\lambda_n) \\ &= \int_\Omega \left(\int_{\sigma(H_0)} a_0(\lambda_0, \omega) dE_0(\lambda_0) \right) V_1 \cdots V_n \left(\int_{\sigma(H_n)} a_n(\lambda_n, \omega) dE_n(\lambda_n) \right) d\nu(\omega) \\ &= \int_\Omega a_0(H_0, \omega) V_1 a_1(H_1, \omega) \cdots V_n a_n(H_n, \omega) d\nu(\omega), \end{aligned}$$

if the right-hand side is defined. Note that this operator will only depend on ϕ, not on the representation in terms of functions a_j!

Bounded MOIs

Peller (2005), Birman–Solomyak (1967)

If $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ is such that $\phi \in \mathfrak{BS}$, which means that

$$\phi(\lambda_0, \dots, \lambda_n) = \int_\Omega a_0(\lambda_0, \omega) \cdots a_n(\lambda_n, \omega) d\nu(\omega);$$

$$\int_\Omega \|a_0(\cdot, \omega)\|_\infty \cdots \|a_n(\cdot, \omega)\|_\infty d|\nu|(\omega) < \infty,$$

then for $V_1, \dots, V_n \in B(\mathcal{H})$ and self-adjoint operators $H_0, \dots, H_n$

$$\psi \mapsto \int_\Omega a_0(H_0, \omega) V_1 a_1(H_1, \omega) \cdots V_n a_n(H_n, \omega) \psi d\nu(\omega), \quad \psi \in \mathcal{H},$$

is a well-defined bounded operator on $\mathcal{H}$. Hence for such ϕ,

$$T_\phi^{H_0, \dots, H_n} : B(\mathcal{H}) \times \cdots \times B(\mathcal{H}) \to B(\mathcal{H}).$$

Commutators

MOIs come with many useful identities, which can then be applied in various contexts. An example:

$$\begin{aligned}(z-H)^{-1}V &= V(z-H)^{-1} + [(z-H)^{-1}, V] \\ &= V(z-H)^{-1} + (z-H)^{-1}[H,V](z-H)^{-1}.\end{aligned}$$

As before, f is holomorphic, taking a contour integral we can write

$$T^H_{f^{[n]}}(V_1,\ldots,V_n) = \frac{1}{2\pi i}\int_\Gamma f(z)(z-H)^{-1}V_1(z-H)^{-1}\cdots V_n(z-H)^{-1}dz,$$

and therefore, *if* $[H,a]$ *is a bounded operator*,

$$\begin{aligned}T^H_{f^{[n]}}(aV_1, V_2, \ldots V_n) = aT^H_{f^{[n]}}(V_1, V_2, \ldots, V_n) \\ + T^H_{f^{[n+1]}}([H,a], V_1, V_2, \ldots, V_n).\end{aligned}$$

This formula holds for non-holomorphic f too.

A slice of life

In NCG, the JLO cocycle plays a role in the proof of the Connes–Moscovici local index theorem, and is defined for $a_0, \ldots, a_n \in B(\mathcal{H})$, n even, as

$$\Psi_n(a_0, \ldots, a_n) = \mathrm{Tr}\left(\eta a_0 \int_{\Delta_n} e^{-t_0 D^2}[D, a_1]e^{-t_1 D^2} \cdots [D, a_n]e^{-t_n D^2} dt\right).$$

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Figure: by Teun van Nuland

JLO as MOI

From the MOI perspective, we can write the JLO cocycle as

$$\mathrm{Tr}\left(\eta a_0 \int_{\Delta_n} e^{-t_0 D^2}[D, a_1]e^{-t_1 D^2} \cdots [D, a_n]e^{-t_n D^2} dt\right) \\ = \mathrm{Tr}(\eta a_0 T^{D^2}_{f^{[n]}}([D, a_1], \dots, [D, a_n])),$$

with $f(x) = \exp(-x)$. Using this observation we can greatly simplify some proofs of ...

Local index formula

The Connes–Moscovici local index formula is a generalisation of the Atiyah–Singer index theorem to noncommutative geometry. We write $V^{(m)} = [D^2, [D^2, [\cdots, [D^2, V]\cdots]]$.

Connes–Moscovici

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple (plus technical conditions). For n odd,

$$\begin{aligned} &\phi_n(a_0, \ldots, a_n) \qquad a_0, \ldots, a_n \in \mathcal{A} \\ &= \sum_{|k|, q \geq 0} c_{n,k,q} \operatorname{Res}_{z=0} z^q \mathrm{Tr}\left(a_0 [D, a_1]^{(k_1)} \cdots [D, a_n]^{(k_n)} |D|^{-2|k|-2z-n}\right) \end{aligned}$$

defines a (b, B)-cocycle whose cohomology class in $HC^{\mathrm{odd}}(\mathcal{A})$ coincides with the cyclic cohomology Chern character $ch_*(\mathcal{A}, \mathcal{H}, D)$.

Problem

There is just one problem, however.

The intermediate steps would involve expressions like

$$T^{D^2}_{f^{[n]}}([D^2, V_1], V_2, \dots, V_n),$$

where $V_1, \dots, V_n \in B(\mathcal{H})$, but $[D^2, V_1]$ is an unbounded operator!

Part 2: Connes–Moscovici's pseudodifferential calculus

Sobolev spaces

Given a densely defined, invertible self-adjoint operator Θ on a Hilbert space $\mathcal{H}$, we can define the 'Sobolev' spaces $\mathcal{H}^s$, $s \in \mathbb{R}$, as the completion of $\operatorname{dom} \Theta^s$ under the norm

$$\|\xi\|_s^2 = \langle \xi, \xi \rangle_s := \langle \Theta^s \xi, \Theta^s \xi \rangle_{\mathcal{H}} = \|\Theta^s \xi\|^2, \quad \xi \in \operatorname{dom} \Theta^s.$$

This forms a Hilbert space. It follows from the assumptions that

$$\mathcal{H}^\infty := \bigcap_{s \in \mathbb{R}} \mathcal{H}^s$$

is dense in $\mathcal{H}$.
We have continuous embeddings

$$\mathcal{H}^t \subseteq \mathcal{H}^s, \quad s \leq t,$$

because

$$\|\Theta^s \xi\| \leq \|\Theta^{s-t}\|_\infty \|\Theta^t \xi\|.$$

Analytic order

Even though Θ itself is an unbounded operator on $\mathcal{H}$, if we regard it as an operator

$$\Theta : \mathcal{H}^1 \to \mathcal{H}^0 = \mathcal{H},$$

it is a perfectly good bounded operator:

$$\|\Theta\|_{\mathcal{H}^1 \to \mathcal{H}^0} = \sup_{\xi : \|\Theta\xi\| \leq 1} \|\Theta\xi\| = 1.$$

We can define $\mathrm{op}^r(\Theta)$ for $r \in \mathbb{R}$ as those operators T on $\mathcal{H}$ such that $\mathcal{H}^\infty \subseteq \mathrm{dom}\, T$, $T\mathcal{H}^\infty \subseteq \mathcal{H}^\infty$, and T extends to a bounded operator

$$T : \mathcal{H}^{s+r} \to \mathcal{H}^s, \quad s \in \mathbb{R}.$$

Note that $\mathrm{op}^r(\Theta) \subseteq \mathrm{op}^t(\Theta)$ for $r \leq t$, and $\mathrm{op}^r(\Theta) \cdot \mathrm{op}^t(\Theta) \subseteq \mathrm{op}^{r+t}(\Theta)$.

Examples

- In a classical setting, if Δ is the Laplace operator on the Euclidean space $\mathbb{R}^n$, setting $\Theta = (1+\Delta)^{1/2}$ precisely gives the classical Sobolev spaces

$$\mathcal{H}^{s,2}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}\big[(1+|\xi|^2)^{s/2}\mathcal{F}f\big] \in L_2(\mathbb{R}^n)\},$$

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  The k-th order (pseudo)differential operators are contained in $\mathrm{op}^k(\Theta)$.
- If Θ is itself a bounded operator on $\mathcal{H}$, then $\mathcal{H}^s \simeq \mathcal{H}$ for all s, and $\mathrm{op}^r(\Theta) = B(\mathcal{H})$ for all r.
- In noncommutative geometry, one has a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, and one usually takes $\Theta = (1+D^2)^{1/2}$. Then for example $D \in \mathrm{op}^1(\Theta)$, and for a *regular* spectral triple $a, [D,a] \in \mathrm{op}^0(\Theta)$ for all $a \in \mathcal{A}$.

Part 3: MOIs as pseudodifferential operators

Unbounded MOIs

Suppose we have a function

$$\phi(\lambda_0, \ldots, \lambda_n) = \int_\Omega a_0(\lambda_0, \omega) \cdots a_n(\lambda_n, \omega) d\nu(\omega).$$

How can we make sense of

$$T_\phi^{H_0, \ldots, H_n}(V_1, \ldots, V_n) = \int_\Omega a_0(H_0, \omega) V_1 a_1(H_1, \omega) \cdots V_n a_n(H_n, \omega) d\nu(\omega),$$

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when $V_i \in \mathrm{op}^{r_i}(\Theta)$?

Observe: if $a_j(H_j, \omega) \in \mathrm{op}^0(\Theta)$, the integrand is a bounded operator $\mathcal{H}^{s+r_1+\cdots+r_n} \to \mathcal{H}^s$ for each $s \in \mathbb{R}$.

Main construction

H., McDonald, van Nuland, Sukochev, Zanin (WIP)

Let $H_0, \ldots, H_n$ be self-adjoint operators on $\mathcal{H}$ with spectral measures dE_j. Let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$. For $V_i \in \mathrm{op}^{r_i}(\Theta)$,

$$T_{\phi}^{H_0,\ldots,H_n}(V_1, \ldots, V_n) = \int \phi(\lambda_0, \ldots, \lambda_n) dE_0(\lambda_0) V_1 dE_1(\lambda_1) \cdots V_n dE_n(\lambda_n),$$

is a well-defined operator

1. in $\mathrm{op}^{r_1+\cdots+r_n}(\Theta)$ if $H_j, [\Theta, H_j] \in \mathrm{op}^{h_j}(\Theta)$ and $\phi \in \mathfrak{BS}^{\infty}$;
2. in $\mathrm{op}^{r_1+\cdots+r_n}(\Theta)$ if each H_j strongly commutes with Θ and $\phi \in \mathfrak{BS}$;
3. in $\mathrm{op}^{r_1+\cdots+r_n+n\frac{s}{2}}(\Theta)$ if each H_j strongly commutes with Θ, $\Theta^{-1} \in \mathcal{L}_s$ for some $0 < s < \infty$, and $\phi \in L_\infty$.

Cases 2 and 3

A quick way to prove cases 2 and 3, is by writing for $V_i \in \mathrm{op}^{r_i}(\Theta)$

$$T_\phi^{H_0,\ldots,H_n}(V_1,\ldots,V_n) = T_\phi^{H_0,\ldots,H_n}(V_1\Theta^{-r_1}\Theta^{r_1}, V_2,\ldots, V_n).$$

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On the RHS, the MOI has bounded arguments.

Case 3 works similarly, if $\Theta^{-1} \in \mathcal{L}_s$, then $\Theta^{-s/2} \in \mathcal{L}_2$ and we can employ the MOI definition for Hilbert–Schmidt operators.

Unbounded symbols

It is possible to also include unbounded symbols ϕ.

H., McDonald, van Nuland, Sukochev, Zanin (WIP)

Let $H_0, \dots, H_n$ be self-adjoint operators in $\mathrm{op}^{h_j}(\Theta)$. For $V_i \in \mathrm{op}^{r_i}(\Theta)$, $P(\lambda_0, \dots, \lambda_n) = \lambda_0^{k_0} \cdots \lambda_n^{k_n}$, $k \in \mathbb{Z}_{\geq 0}$ in each case listed before,

$$T_{P\phi}^{H_0,\dots,H_n}(V_1, \dots, V_n) = T_{\phi}^{H_0,\dots,H_n}(H_0^{k_0} V_1 H_1^{k_1}, \dots, V_n H_n^{k_n})$$

extends the constructions of the MOI given before.

Asymptotic expansion

By applying identities like

$$T^{D^2}_{f^{[n]}}(V_1, V_2, \dots V_n) = V_1 T^{D^2}_{f^{[n]}}(1, V_2, \dots, V_n) + T^{D^2}_{f^{[n+1]}}([D^2, V_1], 1, V_2, \dots, V_n).$$

a million times (which is now possible!), one gets the formal expression (using multi-index notation and $V^{(m)} = [D^2, [\cdots, [D^2, V] \cdots])$

$$T^{(tD)^2}_{f^{[n]}}(V_1, \dots, V_n) \sim \sum_{|m|=0}^{\infty} t^{2|m|} C_m V_1^{(m_1)} \cdots V_n^{(m_n)} T^{(tD)^2}_{f^{[n+|m|]}}(1, \dots, 1)$$
$$= \sum_{|m|=0}^{\infty} t^{2|m|} \frac{C_m}{(n+|m|)!} V_1^{(m_1)} \cdots V_n^{(m_n)} f^{(n+|m|)}(t^2 D^2).$$

Of course, this only makes sense if we can say something about the remainder terms.

Local index formula

This expansion demystifies the local index formula

$$\phi_n(a_0, \dots, a_n) = \sum_{|k|, q \geq 0} c_{n,k,q} \operatorname{Res}_{z=0} z^q \mathrm{Tr} \left(a_0 [D, a_1]^{(k_1)} \cdots [D, a_n]^{(k_n)} |D|^{-2|k|-2z-n} \right)$$

as being related to the expansion of

$$a_0 T^{D^2}_{f_n^{[n]}}([D, a_1], \dots, [D, a_n])$$

for $f_n(x) = x^{n/2}$, but can also be used to prove new results.

Existence of asymptotic expansions

H., McDonald, van Nuland, Sukochev, Zanin (WIP)

Let $(\mathcal{A}, \mathcal{H}, D)$ be an s-summable spectral triple (i.e. $(1 + D^2)^{-1/2} \in \mathcal{L}_s$, $s > 0$) and denote the algebra of operators generated by $\mathcal{A}$ and D by $\mathcal{B}$. Let $V \in \mathcal{B}$ be self-adjoint and bounded. If $\mathrm{Tr}(Pe^{-t^2D^2})$ admits an asymptotic expansion as $t \to 0$ for each operator $P \in \mathcal{B}$, then

$$\mathrm{Tr}(Pe^{-t^2(D+V)^2})$$

also admits an asymptotic expansion as $t \to 0$ for each $P \in \mathcal{B}$, given by

$$\mathrm{Tr}(Pe^{-t^2(D+V)^2}) \sim \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{|m|=0}^{\infty} \frac{(-1)^{n+|m|}}{(n+|m|)!} t^{2(n+|m|)-1} c_{n,k} C_m \times \mathrm{Tr}\big(PD_{0,k} V^{(m_1)} D_{1,k} \cdots V^{(m_n)} D_{n,k} e^{-t^2D^2}\big).$$

Thanks

Thank you for your attention!