The noncommutative space we live in

Postgraduate Conference 2023: Research translation in all its forms

Eva-Maria Hekkelman
PhD supervisors: Fedor Sukochev,
Ed McDonald (Penn State) and Dmitriy Zanin

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Ingredients of this talk

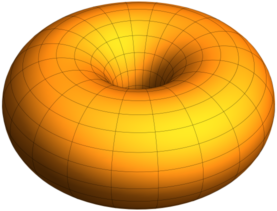
1. Essence of noncommutative geometry
2. A hint of physics
3. Multiple operator integrals

This talk is based on work in progress with Teun van Nuland, Fedor Sukochev and Dmitriy Zanin.

Part 1: Noncommutative Geometry

Differential geometry

Imagine that we are interested in some geometric object X.



We can study this as a *Riemannian manifold*, which is to say a (compact) topological space X where we keep track of things like:

- The distance between points in the space;
- The curvature at each point;
- A sense of 'straight lines' (geodesics).

Algebraic geometry?

Let us take a different perspective, considering instead the continuous functions

$$f : X \to \mathbb{C}.$$

We can give this the structure of an algebra, with the pointwise operations

$$\begin{aligned}(f+g)(x) &= f(x) + g(x);\\ (fg)(x) &= f(x)g(x), \quad x \in X.\end{aligned}$$

With the norm $\|f\| = \sup_{x\in X} |f(x)|$ and involution $f^*(x) = \overline{f(x)}$ this forms a unital commutative C^*-algebra.

Other examples of C^*-algebras are the matrices $M_n(\mathbb{C})$ and $B(H)$, the bounded operators on a Hilbert space.

Translations

If we have a subset $A \subset X$, then

$$\{f \in C(X) : f|_A \equiv 0\}$$

forms an ideal in $C(X)$. In fact, we have the following translations:

- Subsets $\Rightarrow$ ideals;
- Points $\Rightarrow$ maximal ideals;
- Connected components $\Rightarrow$ projections (read: a function that only takes values 0 and 1).

And many more!

Gelfand duality

In fact, this connection is very strong.

Gelfand duality (1940s)

Every unital commutative C^*-algebra A is of the form $A = C(X)$ for a compact Hausdorff space X, and this correspondence is one-to-one. (Even better, this is a dual equivalence of categories.)

This means that *all* topological data of X is contained in $C(X)$. But we lose all geometric data.

A different approach

Can one hear the shape of a drum?

Figure: Mark Kac, Center for Nonlinear Studies.

Hearing the shape of a drum

The sounds a drum produces are those $\lambda \in \mathbb{C}$ for which the PDE

$$\begin{cases} \Delta u & = \lambda u \quad \text{on } X; \\ u|_{\partial X} & = 0 \end{cases}$$

has a solution (Helmholtz equation). Here Δ is the Laplace–Beltrami operator, the manifold equivalent of the differential operator $-\sum_{j=1}^{n} \partial_{x_j}^2$. This is an unbounded operator on the Hilbert space

$$L_2(X) = \{f : X \to \mathbb{C} : \int_X |f(x)|^2 d\mathrm{vol}(x) < \infty\},$$

and these λ are the eigenvalues of Δ.

The question asks whether we can reconstruct our Riemannian manifold X from the data $(L_2(X), \Delta)$, in particular, from these eigenvalues of the operator Δ (this is called *spectral geometry*).

What we can hear

Weyl's law (1910s)

Let X be a compact Riemannian manifold of dimension d. Let $N(t)$ be the number of eigenvalues (counting multiplicities) of Δ with absolute value less than t. Then as $t \to \infty$,

$$N(t) = \frac{\omega_d}{(2\pi)^d}\,\mathrm{Vol}(X)t^{d/2} - \frac{\omega_{d-1}}{4(2\pi)^{d-1}}\,\mathrm{Area}(\partial X)t^{(d-1)/2} + o(t^{(d-1)/2}).$$

What we cannot hear

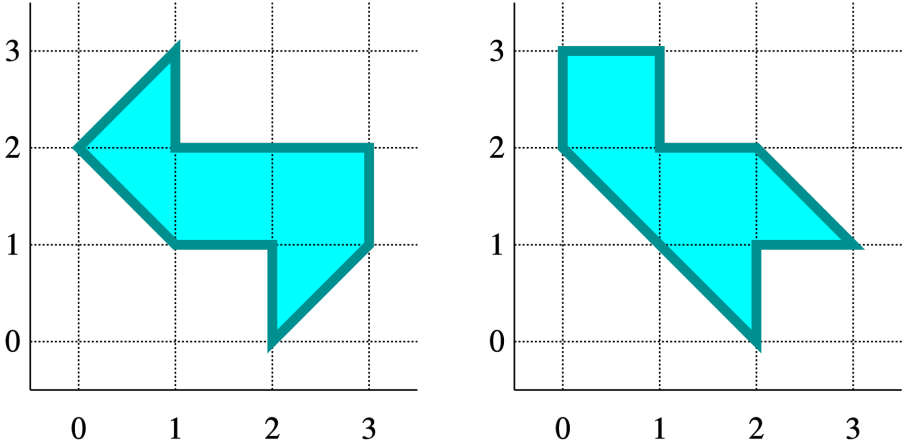


Figure: Isospectral drums.

Spectral triples

$C(X)$ gives us topological data, but no geometrical data. For $(L_2(X), \Delta)$ it is the other way around.

Combined, the triple $(C(X), L_2(X), \Delta)$ seems promising!

We can make these data 'talk' to each other by representing $C(X)$ as operators on the Hilbert space $L_2(X)$ by the representation

$$\begin{aligned} \pi(f) : L_2(X) &\to L_2(X) \\ \xi &\mapsto f \cdot \xi. \end{aligned}$$

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For technical reasons, it is more convenient to replace $C(X)$ by $C^\infty(X)$, Δ by its square root D, the Dirac operator, and the Hilbert space $L_2(X)$ by $L_2(X, S)$ where $S \to X$ is a vector bundle (spinor bundle).
The triple $(C^\infty(X), L_2(X, S), D)$ is an example of a *spectral triple*.

Spectral reconstruction

This combination recovers distances between points:

$$d(x,y) = \sup_{f \in C^\infty(X)} \{|f(x) - f(y)| : \|[D, \pi(f)]\| \leq 1\}.$$

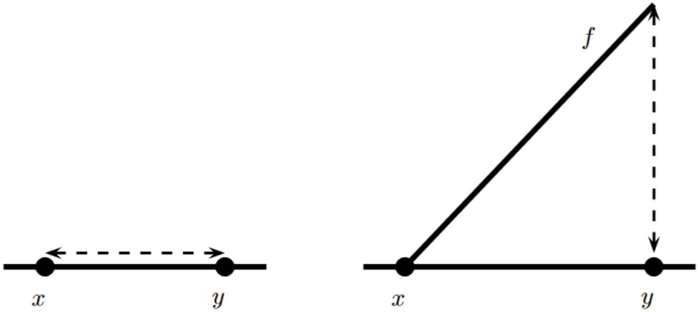

Figure: Van Suijlekom, W. D. (2015). *Noncommutative geometry and particle physics*. Dordrecht: Springer.

Connes reconstruction theorem

Connes (2008, conj. 1996).

For any commutative $*$-algebra A, Hilbert space H and self-adjoint operator D such that (A, H, D) forms a d-dimensional commutative spectral triple, there is a Riemannian manifold X, a vector bundle $S \to X$ and a Dirac-type operator D_S such that

$$(A, H, D) = (C^\infty(X), L_2(X, S), D_S).$$

Furthermore, this association is unique (up to certain natural isomorphisms).

Noncommutative geometry

So far, we have been doing "algebraic geometry", capturing the Riemannian manifold entirely in the data (A, H, D), where:

- A is a commutative $*$-algebra, H a Hilbert space and D an unbounded self-adjoint operator on H;
- A is represented as bounded operators on H;
- Some technical compatibility conditions.

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Noncommutative geometry is the study of such triples (A, H, D), where we drop the requirement that A is commutative.

Part 2: The physicsy bit

Standard model

$$\mathcal{L}_{SM} = -\frac{1}{2}\partial_\nu g^a_\mu \partial_\nu g^a_\mu - g_s f^{abc}\partial_\mu g^a_\nu g^b_\mu g^c_\nu - \frac{1}{4}g_s^2 f^{abc} f^{ade} g^b_\mu g^c_\nu g^d_\mu g^e_\nu + \frac{1}{2}ig_s^2(\bar{q}^\sigma_i \gamma^\mu q^\sigma_j) g^a_\mu + \bar{G}^a \partial^2 G^a + g_s f^{abc}\partial_\mu \bar{G}^a G^b g^c_\mu - \partial_\nu W^+_\mu \partial_\nu W^-_\mu - M^2 W^+_\mu W^-_\mu - \frac{1}{2}\partial_\nu Z^0_\mu \partial_\nu Z^0_\mu - \frac{1}{2c_w^2} M^2 Z^0_\mu Z^0_\mu - \frac{1}{2}\partial_\mu A_\nu \partial_\mu A_\nu - \frac{1}{2}\partial_\mu H \partial_\mu H - \frac{1}{2}m_h^2 H^2 - \partial_\mu \phi^+ \partial_\mu \phi^- - M^2 \phi^+ \phi^- - \frac{1}{2}\partial_\mu \phi^0 \partial_\mu \phi^0 - \frac{1}{2c_w^2} M \phi^0 \phi^0 - \beta_h[\frac{2M^2}{g^2} + \frac{2M}{g} H + \frac{1}{2}(H^2 + \phi^0\phi^0 + 2\phi^+\phi^-)] + \frac{2M^4}{g^2}\alpha_h - igc_w[\partial_\nu Z^0_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - Z^0_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + Z^0_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)] - igs_w[\partial_\nu A_\mu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - A_\nu (W^+_\mu \partial_\nu W^-_\mu - W^-_\mu \partial_\nu W^+_\mu) + A_\mu (W^+_\nu \partial_\nu W^-_\mu - W^-_\nu \partial_\nu W^+_\mu)] - \frac{1}{2}g^2 W^+_\mu W^-_\mu W^+_\nu W^-_\nu + \frac{1}{2}g^2 W^+_\mu W^-_\nu W^+_\mu W^-_\nu + g^2 c_w^2 (Z^0_\mu W^+_\mu Z^0_\nu W^-_\nu - Z^0_\mu Z^0_\mu W^+_\nu W^-_\nu) + g^2 s_w^2 (A_\mu W^+_\mu A_\nu W^-_\nu - A_\mu A_\mu W^+_\nu W^-_\nu) + g^2 s_w c_w [A_\mu Z^0_\nu (W^+_\mu W^-_\nu - W^+_\nu W^-_\mu) - 2A_\mu Z^0_\mu W^+_\nu W^-_\nu] - g\alpha[H^3 + H\phi^0\phi^0 + 2H\phi^+\phi^-] - \frac{1}{8}g^2\alpha_h[H^4 + (\phi^0)^4 + 4(\phi^+\phi^-)^2 + 4(\phi^0)^2\phi^+\phi^- + 4H^2\phi^+\phi^- + 2(\phi^0)^2 H^2] - gMW^+_\mu W^-_\mu H - \frac{1}{2}g\frac{M}{c_w^2} Z^0_\mu Z^0_\mu H - \frac{1}{2}ig[W^+_\mu(\phi^0\partial_\mu\phi^- - \phi^-\partial_\mu\phi^0) - W^-_\mu(\phi^0\partial_\mu\phi^+ - \phi^+\partial_\mu\phi^0)] + \frac{1}{2}g[W^+_\mu(H\partial_\mu\phi^- - \phi^-\partial_\mu H) - W^-_\mu(H\partial_\mu\phi^+ - \phi^+\partial_\mu H)] + \frac{1}{2}g\frac{1}{c_w}(Z^0_\mu(H\partial_\mu\phi^0 -$$

The noncommutative standard model

If we augment the commutative spectral triple $(C^\infty(X), L_2(X,S), D_S)$ with the carefully chosen data (A_F, H_F, D_F), putting $D := D_S \otimes 1 + \gamma \otimes D_F$, we get the (noncommutative) spectral triple

$$(C^\infty(X) \otimes A_F, L_2(X,S) \otimes H_F, D).$$

Perturbing D by suitable self-adjoint operators V, the functional

$$f \mapsto \mathrm{Tr}(f(D+V)), \quad f : \mathbb{R} \to \mathbb{R}_{\geq 0} \text{ even, s.t. } f(D) \in \mathcal{L}_1,$$

recovers the entire Lagrangian of the Standard Model with some coefficients depending on f. This is called the *spectral action*.

Part 3: Research

Functional calculus

It seems important to really understand $\mathrm{Tr}(f(D+V))$ for self-adjoint operators D and V.

For a bounded operator A on a Hilbert space, the operator $\exp(tA)$ can be defined as a limit of $\sum_{n\leq N} \frac{t^n}{n!} A^n$ (think of your ODE course).

We can go further and define $f(D)$ for a.e.-finite measurable functions $f : \mathbb{R} \to \mathbb{C}$ and (unbounded) self-adjoint operators D.

Taylor expansion

If f is smooth enough, a naive way to analyse $\mathrm{Tr}(f(D+V))$ is to do a Taylor expansion of the function $t \mapsto \mathrm{Tr}(f(D+tV))$ around $t=0$:

$$\begin{aligned}\mathrm{Tr}(f(D+V)) &=^{?} \sum_{n=0}^{\infty} \frac{1}{n!} \frac{d^n}{dt^n}\bigg|_{t=0} \mathrm{Tr}(f(D+tV)) \\ &=^{!?} \sum_{n=0}^{\infty} \mathrm{Tr}\left(\frac{1}{n!} \frac{d^n}{dt^n}\bigg|_{t=0} f(D+tV) \right).\end{aligned}$$

What on earth is $\frac{1}{n!}\frac{d^n}{dt^n}\Big|_{t=0} f(D+tV)$?

Multiple operator integrals

If a function $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ admits a representation of the form

$$\phi(\lambda_0, \dots, \lambda_n) = \int_\Omega g_0(\lambda_0, \omega) \cdots g_n(\lambda_n, \omega) d\nu(\omega),$$

(with some condition on the g_j) and self-adjoint operators $H_0, \dots, H_n$, we can define the transformer

$$\begin{aligned} T_\phi^{H_0,\dots,H_n} : & B(H) \times \cdots \times B(H) \to B(H) \\ & (V_1, \dots, V_n) \mapsto \int_\Omega g_0(H_0, \omega) V_1 g_1(H_1, \omega) \cdots V_n g_n(H_n, \omega) d\nu(\omega). \end{aligned}$$

This does not depend on how we represent ϕ!

Divided differences

Given a smooth function f, we can define

$$f^{[1]}(\lambda_0, \lambda_1) = \frac{f(\lambda_0) - f(\lambda_1)}{\lambda_0 - \lambda_1}.$$

By the Cauchy integral formula,

$$f^{[1]}(\lambda_0, \lambda_1) = \int_\gamma f(z)(z - \lambda_0)^{-1}(z - \lambda_1)^{-1} dz,$$

but we also have the representation

$$f^{[1]}(\lambda_0, \lambda_1) = \int_{\Delta_1} \int_{\mathbb{R}} \widehat{f'}(t) e^{its_0\lambda_0} e^{its_1\lambda_1} dt d\sigma(s_0, s_1),$$

and so we always have

$$\int_\gamma f(z)(z - H_0)^{-1} V (z - H_1)^{-1} dz = \int_{\Delta_1} \int_{\mathbb{R}} \widehat{f'}(t) e^{its_0 H_0} V e^{its_1 H_1} dt d\sigma(s_0, s_1).$$

Taylor expansions!

We can define higher *divided differences* of f, and then we can make sense of

$$\begin{aligned}\frac{1}{n!}\frac{d^n}{dt^n}\Big|_{t=0} f(D+tV) &= T^{D,\dots,D}_{f^{[n]}}(V,\dots,V)\\ &= \int_\gamma f(z)(z-D)^{-1}V(z-D)^{-1}\cdots V(z-D)^{-1}dz.\end{aligned}$$

Work in progress

Current work with my supervisors and Teun van Nuland:

WIP (2023)

Let $H_0, \ldots, H_n$ be self-adjoint operators and let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ be such that the transformer

$$T_\phi^{H_0,\ldots,H_n} : B(H) \times \cdots \times B(H) \to B(H)$$

is defined. Then (leaving out technical details), $T_\phi^{H_0,\ldots,H_n}$ can be extended to a transformer of unbounded operators

$$T_\phi^{H_0,\ldots,H_n} : \mathrm{op}^{r_1} \times \cdots \mathrm{op}^{r_n} \to \mathrm{op}^{r_1+\cdots+r_n}.$$

Thanks

Thank you for your attention!

Take home messages:

- You can reconstruct a Riemannian manifold given the spectral triple $(C^\infty(X), L_2(X, S), D_S)$;
- Noncommutative geometry is the study of (noncommutative) spectral triples (A, H, D);
- Multiple operator integrals are a powerful technical tool in this field.