Unbounded Multiple Operator Integrals

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Summary of this talk

1. Motivation
2. Pseudodifferential calculus
3. Multiple operator integrals as pseudodifferential operators

This talk is based on work in progress with Teun van Nuland, Fedor Sukochev and Dmitriy Zanin.

Part 1: Motivation

A slice of life

An important object in noncommutative geometry is the JLO cocycle, which is defined for $a_0, \ldots, a_n \in B(\mathcal{H})$, n even, as

$$\Psi_n(a_0, \ldots, a_n) = \mathrm{Tr}\left(\eta a_0 \int_{\Delta_n} e^{-t_0 D^2}[D, a_1]e^{-t_1 D^2} \cdots [D, a_n]e^{-t_n D^2} dt\right).$$

Here Δ_n is the standard n-simplex, and D is an unbounded self-adjoint operator.

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Figure: by Teun van Nuland

Multiple operator integrals

If a function $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ admits a representation of the form

$$\phi(\lambda_0, \ldots, \lambda_n) = \int_\Omega g_0(\lambda_0, \omega) \cdots g_n(\lambda_n, \omega) d\nu(\omega),$$

with (Ω, ν) and $g_j : \mathbb{R} \times \Omega \to \mathbb{C}$ nice enough, then for self-adjoint operators $H_0, \ldots, H_n$, we can define the transformer

$$T_\phi^{H_0,\ldots,H_n} : B(\mathcal{H}) \times \cdots \times B(\mathcal{H}) \to B(\mathcal{H})$$
$$T_\phi^{H_0,\ldots,H_n}(V_1, \ldots, V_n) = \int_\Omega g_0(H_0, \omega) V_1 g_1(H_1, \omega) \cdots V_n g_n(H_n, \omega) d\nu(\omega).$$

This does not depend on how we represent ϕ!

Taylor expansions

An important application of MOIs is to make sense of Taylor expansions of the functional calculus. For $f : \mathbb{R} \to \mathbb{R}$ smooth enough, we recursively define

$$f^{[n]}(\lambda_0, \dots, \lambda_n) = \frac{f^{[n-1]}(\lambda_0, \dots, \lambda_{n-1}) - f^{[n-1]}(\lambda_1, \dots, \lambda_n)}{\lambda_0 - \lambda_n}.$$

In particular,

$$\frac{1}{n!} f^{(n)}(\lambda) = f^{[n]}(\lambda, \dots, \lambda).$$

For densely defined, self-adjoint H and bounded s.-a. V, (abbreviating $T_\phi^{H,\dots,H}$ to T_ϕ^H), and f regular enough,

$$\frac{1}{n!} \frac{d^n}{dt^n}\Big|_{t=0} f(H + tV) = T^H_{f^{[n]}}(V, \dots, V).$$

Commutators

MOIs come with many useful identities, which can then be applied in various contexts. An example:

$$\begin{aligned}(z-H)^{-1}V &= V(z-H)^{-1} + [(z-H)^{-1}, V] \\ &= V(z-H)^{-1} + (z-H)^{-1}[H,V](z-H)^{-1}.\end{aligned}$$

If f is holomorphic, taking a contour integral we can write

$$T^H_{f^{[n]}}(V_1,\dots,V_n) = \frac{1}{2\pi i}\int_\Gamma f(z)(z-H)^{-1}V_1(z-H)^{-1}\cdots V_n(z-H)^{-1}dz,$$

and therefore

$$\begin{aligned}T^H_{f^{[n]}}(aV_1, V_2, \dots V_n) &= aT^H_{f^{[n]}}(V_1, V_2, \dots, V_n) \\ &\quad + T^H_{f^{[n+1]}}([H,a], V_1, V_2, \dots, V_n).\end{aligned}$$

This formula holds for non-holomorphic f too.

JLO as MOI

From this perspective, we can write the JLO cocycle as

$$\mathrm{Tr}\left(\eta a_0 \int_{\Delta_n} e^{-t_0 D^2}[D, a_1]e^{-t_1 D^2} \cdots [D, a_n]e^{-t_n D^2} dt\right)$$
$$= \mathrm{Tr}(\eta a_0 T^{D^2}_{f^{[n]}}([D, a_1], \ldots, [D, a_n])),$$

with $f(x) = \exp(-x)$. Using this observation we can obtain a clean and hassle-free proof of ...

Local index formula

The Connes–Moscovici local index formula is a generalisation of the Atiyah–Singer index theorem to noncommutative geometry. We write $X^{(m)} = [D^2, [D^2, [\cdots, [D^2, X]\cdots]]$.

Connes–Moscovici

Let $(\mathcal{A}, \mathcal{H}, D)$ be a spectral triple (plus technical conditions). For n odd,

$$\begin{aligned} &\phi_n(a_0, \dots, a_n) \qquad a_0, \dots, a_n \in \mathcal{A} \\ &= \sum_{|k|, q \geq 0} c_{n,k,q} \operatorname{Res}_{z=0} z^q \mathrm{Tr}\left(a_0 [D, a_1]^{(k_1)} \cdots [D, a_n]^{(k_n)} |D|^{-2|k|-2z-n}\right) \end{aligned}$$

defines a (b, B)-cocycle whose cohomology class in $HC^{\mathrm{odd}}(\mathcal{A})$ coincides with the cyclic cohomology Chern character $ch_*(\mathcal{A}, \mathcal{H}, D)$.

Problem

There is just one problem, however.

The intermediate steps would involve expressions like

$$T^{D^2}_{f^{[n]}}([D^2, X_1], X_2, \ldots, X_n),$$

where $X_1, \ldots, X_n \in B(\mathcal{H})$, but $[D^2, X_1]$ is an unbounded operator!

Part 2: Connes–Moscovici's pseudodifferential calculus

Sobolev spaces

Given a densely defined, invertible self-adjoint operator Θ on a Hilbert space $\mathcal{H}$, we can define the 'Sobolev' spaces $\mathcal{H}^s$, $s \in \mathbb{R}$, as the completion of $\operatorname{dom} \Theta^s$ under the norm

$$\|\xi\|_s^2 = \langle \xi, \xi \rangle_s := \langle \Theta^s \xi, \Theta^s \xi \rangle_{\mathcal{H}} = \|\Theta^s \xi\|^2, \quad \xi \in \operatorname{dom} \Theta^s.$$

This forms a Hilbert space. We will assume that

$$\mathcal{H}^\infty := \bigcap_{s \in \mathbb{R}} \mathcal{H}^s$$

is dense in $\mathcal{H}$.
We have continuous embeddings

$$\mathcal{H}^t \subseteq \mathcal{H}^s, \quad s \leq t,$$

because

$$\|\Theta^s \xi\| \leq \|\Theta^{s-t}\|_\infty \|\Theta^t \xi\|.$$

Analytic order

Even though Θ itself is an unbounded operator on $\mathcal{H}$, if we regard it as an operator

$$\Theta : \mathcal{H}^1 \to \mathcal{H}^0 = \mathcal{H},$$

it is a perfectly good bounded operator:

$$\|\Theta\|_{\mathcal{H}^1 \to \mathcal{H}^0} = \sup_{\xi : \|\Theta\xi\| \leq 1} \|\Theta\xi\| = 1.$$

We can define op^r for $r \in \mathbb{R}$ as those operators T on $\mathcal{H}$ such that $\mathcal{H}^\infty \subseteq \mathrm{dom}\, T$, $T\mathcal{H}^\infty \subseteq \mathcal{H}^\infty$, and T extends to a bounded operator

$$T : \mathcal{H}^{s+r} \to \mathcal{H}^s, \quad s \in \mathbb{R}.$$

Note that $\mathrm{op}^r \subseteq \mathrm{op}^t$ for $r \leq t$.

Examples

- In a classical setting, if Δ is the Laplace operator on the Euclidean space $\mathbb{R}^n$, setting $\Theta = (1+\Delta)^{1/2}$ precisely gives the classical Sobolev spaces

$$\mathcal{H}^{s,2}(\mathbb{R}^n) := \{f \in \mathcal{S}'(\mathbb{R}^n) : \mathcal{F}^{-1}\left[(1+|\xi|^2)^{s/2}\mathcal{F}f\right] \in L_2(\mathbb{R}^n)\},$$

where $\mathcal{F}$ is the Fourier transform.
The k-th order (pseudo)differential operators are contained in op^k, as are the Fourier multipliers T_m for which $m(\xi) = O(|\xi|^k)$.

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  where $\mathcal{F}$ is the Fourier transform.
  The k-th order (pseudo)differential operators are contained in op^k, as are the Fourier multipliers T_m for which $m(\xi) = O(|\xi|^k)$.
- If Θ is itself a bounded operator on $\mathcal{H}$, then $\mathcal{H}^s \simeq \mathcal{H}$ for all s, and $\mathrm{op}^r = B(\mathcal{H})$ for all r.
- In noncommutative geometry, one has a spectral triple $(\mathcal{A}, \mathcal{H}, D)$, and one usually takes $\Theta = (1+D^2)^{1/2}$. Then for example $D \in \mathrm{op}^1$, and for a *regular* spectral triple $a, [D, a] \in \mathrm{op}^0$ for all $a \in \mathcal{A}$.

Part 3: MOIs as pseudodifferential operators

Unbounded MOIs

Suppose we have a function

$$\phi(\lambda_0, \ldots, \lambda_n) = \int_\Omega g_0(\lambda_0, \omega) \cdots g_n(\lambda_n, \omega) d\nu(\omega)$$

such that for self-adjoint $H_0, \ldots, H_n$ and $V_1, \ldots, V_n \in B(\mathcal{H})$

$$T_\phi^{H_0,\ldots,H_n}(V_1, \ldots, V_n) = \int_\Omega g_0(H_0, \omega) V_1 g_1(H_1, \omega) \cdots V_n g_n(H_n, \omega) d\nu(\omega)$$

defines a bounded operator.

If $V_i \in \mathrm{op}^{r_i}$, and $g_j(H_j, \omega) \in \mathrm{op}^0$ the integrand is unbounded on $\mathcal{H}$, but can be considered a bounded operator $\mathcal{H}^{s+r_1+\cdots+r_n} \to \mathcal{H}^s$ for each $s \in \mathbb{R}$.

MOI as ΨDO

Given a pseudodifferential calculus generated by a densely defined, invertible self-adjoint operator Θ on a Hilbert space $\mathcal{H}$, we can define MOIs as follows.

H., van Nuland, Sukochev, Zanin (2023, WIP)

Let $H_0, \dots, H_n$ be self-adjoint operators that strongly commute with Θ. Let $\phi : \mathbb{R}^{n+1} \to \mathbb{C}$ be in the Birman–Solomyak function class. Then for $X_i \in \mathrm{op}^{r_i}$,

$$T_\phi^{H_0,\dots,H_n}(X_1,\dots,X_n) = \int_\Omega a_0(H_0,\omega)X_1 a_1(H_1,\omega)\cdots X_n a_n(H_n,\omega)d\nu(\omega) \tag{1}$$

is a well-defined operator in $\mathrm{op}^{r_1+\cdots+r_n}$.

Relation to bounded MOIs

A quick way to see that this works, is by writing for $X_i \in \mathrm{op}^{r_i}$

$$T_\phi^{H_0,\ldots,H_n}(X_1,\ldots,X_n) = T_\phi^{H_0,\ldots,H_n}(X_1\Theta^{-r_1}\Theta^{r_1}, X_2, \ldots, X_n).$$

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On the RHS, the MOI has bounded arguments.

This also shows that unbounded MOIs inherit many properties from usual MOI theory.

To emphasise: this paradigm of MOIs makes sense of multiple operator integrals whose arguments are differential operators, pseudodifferential operators or Fourier multipliers.

Asymptotic expansion

By applying identities like

$$T^{D^2}_{f^{[n]}}(X_1, X_2, \dots X_n) = X_1 \, T^{D^2}_{f^{[n]}}(1, X_2, \dots, X_n) + T^{D^2}_{f^{[n+1]}}([D^2, X_1], 1, X_2, \dots, X_n).$$

a million times (which is now possible!), one gets the formal expression (using multi-index notation and $X^{(m)} = [D^2, [\cdots, [D^2, X] \cdots])$

$$T^{(tD)^2}_{f^{[n]}}(X_1, \dots, X_n) \sim \sum_{|m|=0}^{\infty} t^{2|m|} C_m X_1^{(m_1)} \cdots X_n^{(m_n)} \, T^{(tD)^2}_{f^{[n+|m|]}}(1, \dots, 1)$$

$$= \sum_{|m|=0}^{\infty} t^{2|m|} \frac{C_m}{(n+|m|)!} X_1^{(m_1)} \cdots X_n^{(m_n)} f^{(n+|m|)}(t^2 D^2).$$

Of course, this only makes sense if we can say something about the remainder terms.

Local index formula

This expansion demystifies the local index formula

$$\phi_n(a_0, \ldots, a_n)$$
$$= \sum_{|k|,q\geq 0} c_{n,k,q} \operatorname{Res}_{z=0} z^q \mathrm{Tr}\left(a_0[D,a_1]^{(k_1)} \cdots [D,a_n]^{(k_n)}|D|^{-2|k|-2z-n}\right)$$

as being related to the expansion of

$$a_0 T^{D^2}_{f_n^{[n]}}([D,a_1], \ldots, [D,a_n])$$

for $f_n(x) = x^{n/2}$, but can also be used to prove new results.

Existence of asymptotic expansions

H., van Nuland, Sukochev, Zanin (2023, WIP)

Let $\mathcal{A}$ be an algebra of bounded operators and let D be a densely defined, self-adjoint operator s.t. $(D-i)^{-1} \in \mathcal{L}_s$, $s>0$, (for example, $(\mathcal{A}, \mathcal{H}, D)$ is s-summable spectral triple) and denote the algebra of operators generated by $\mathcal{A}$ and D by $\mathcal{B}$. Let $P, V \in \mathcal{B}$ with V self-adjoint and bounded. If $\mathrm{Tr}(Qe^{-t^2D^2})$ admits an asymptotic expansion as $t \to 0$ for each operator $Q \in \mathcal{B}$, then

$$\mathrm{Tr}(Pe^{-t^2(D+V)^2})$$

also admits an asymptotic expansion as $t \to 0$, given by

$$\mathrm{Tr}(Pe^{-t^2(D+V)^2}) \sim \sum_{n=0}^{\infty} \sum_{k=0}^{n} \sum_{|m|=0}^{\infty} \frac{(-1)^{n+|m|}}{(n+|m|)!} t^{2(n+|m|)-1} c_{n,k} C_m \times \mathrm{Tr}\big(PD_{0,k} V^{(m_1)} D_{1,k} \cdots V^{(m_n)} D_{n,k} e^{-t^2D^2}\big).$$

Thanks

Thank you for your attention!